LINEARISATION OF THE M.S.E. VIA THE LEGENDRE TRANSFORMATION

KIAN SHAH
S5578191

We consider a bundle $\pi : \mathbb{R} \times V \to V$, where $V \subseteq \mathbb{R}^2$. The fiber coordinate is denoted by u and the coordinates on $\mathbb{R}^2$ by x, y. We consider the minimal surface equation for this smooth function $u(x, y)$ defined on the open domain $V \subset \mathbb{R}^2$:

$$\mathcal{E}_{\min \Sigma} = \{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0\}. \tag{1}$$

Viewing this setup geometrically, we're working in the first jet space $J^1(\pi)$, which consists of tuples (x, y, u, u_x, u_y). The Legendre map is defined by:

$$\mathcal{L} : J^1(\pi) \to \mathbb{R}^3, \quad (x, y, u, u_x, u_y) \mapsto (p = u_x,\ q = u_y,\ \Phi = xu_x + yu_y - u).$$

Note that this map is not injective in general: for fixed values of (p, q, Φ), the set of all (x, y, u) satisfying $\Phi = xp + yq - u$ forms a 2-dimesnional affine subspace. This is because we have one equation with 3 unknowns, and so the set of solutions is a 2-dimensional set. Since the pre-image of a point is a 2-dimensional affine space, the map is not injective. To resolve this, we restrict the Legendre map $\mathcal{L}$ to the graph of a solution $u(x, y)$. Define the gradient map:

$$G(x, y) = (u_x(x, y), u_y(x, y)) = (p, q).$$

Assume G is an immersion, i.e., its Jacobian (which is the Hessian matrix of u) is nonsingular at each point. Then, by the inverse function theorem, we can locally invert G to express:

$$p = u_x(x, y), q = u_y(x, y) \implies x = x(p, q), \quad y = y(p, q). \tag{2}$$

This lets us define the function $\Phi(p, q)$ as:

$$\Phi(p, q) := x(p, q)\, p + y(p, q)\, q - u(x(p, q), y(p, q)).$$

We aim to show that if $u(x, y)$ satisfies the nonlinear minimal surface equation (1), then the Legendre transform $\Phi(p, q)$ satisfies the linear second-order PDE:

$$(1 + q^2)\Phi_{qq}(p, q) + 2pq\Phi_{pq}(p, q) + (1 + p^2)\Phi_{pp}(p, q) = 0. \tag{3}$$

We will also show that the converse holds: any sufficiently smooth solution $\Phi(p, q)$ of the above linear PDE gives rise (via inverse Legendre transform) to a function $u(x, y)$ solving the minimal surface equation (1).

Date: October 18, 2026.

We start by differentiating $\Phi(p,q)$ with respect to p and q:

$$\begin{aligned}\Phi_p(p,q) &= x(p,q) + px_p(p,q) + qy_p(p,q)\\ &\quad - u_x(x(p,q),y(p,q))x_p(p,q) - u_y(x(p,q),y(p,q))y_p(p,q),\\ \Phi_q(p,q) &= y(p,q) + px_q(p,q) + qy_q(p,q)\\ &\quad - u_x(x(p,q),y(p,q))x_q(p,q) - u_y(x(p,q),y(p,q))y_q(p,q).\end{aligned}$$

But since $p = u_x(x(p,q),y(p,q))$, $q = u_y(x(p,q),y(p,q))$, this simplifies to:

$$\Phi_p(p,q) = x(p,q), \quad \Phi_q(p,q) = y(p,q). \tag{4}$$

By the equality of mixed derivatives ($\Phi_{pq} = \Phi_{qp}$) we have that:

$$\Phi_{pq}(p,q) = x_q(p,q) = y_p(p,q) = \Phi_{qp}(p,q).$$

We now assume u satisfies the minimal surface equation. We multiply this equation by x_q, giving:

$$\begin{aligned}(1+q^2)u_{xx}(x(p,q),y(p,q))x_q(p,q) - 2pqu_{xy}(x(p,q),y(p,q))x_q(p,q) +\\ (1+p^2)u_{yy}(x(p,q),y(p,q))x_q(p,q) = 0.\end{aligned} \tag{5}$$

To eliminate u_{xx} and u_{yy}, we differentiate $p = u_x(x(p,q),y(p,q))$ and $q = u_y(x(p,q),y(p,q))$ with respect to q and p (respectively) using the chain rule:

$$\begin{aligned}0 = \partial_q p = u_{xx}x_q + u_{xy}y_q \Rightarrow u_{xx}x_q = -u_{xy}y_q,\\ 0 = \partial_p q = u_{xy}x_p + u_{yy}y_p \Rightarrow u_{yy}x_q = -u_{xy}x_p.\end{aligned}$$

Substituting into the equation above:

$$\begin{aligned}-(1+q^2)u_{xy}(x(p,q),y(p,q))y_q(p,q) - 2pqu_{xy}(x(p,q),y(p,q))x_q(p,q)\\ -(1+p^2)u_{xy}(x(p,q),y(p,q))x_p(p,q) = 0.\end{aligned} \tag{6}$$

It seems logical now to divide by the common factor $u_{xy}(x(p,q),y(p,q))$ in equation (6), however, since this term may be zero, we need to work carefully. We define the closed set C_u associated to the function $u(x,y)$

$$C_u = \{(x,y) : u_{xy}(x,y) = 0\},$$

and distinguish the following cases:

- Case 1: $(x(p,q),y(p,q)) \notin C_u$, i.e., $u_{xy}(x(p,q),y(p,q)) \neq 0$.

  In this case, we can safely divide both sides of equation (6) by $u_{xy}(x(p,q),y(p,q))$, giving:

  $$-(1+q^2)y_q(p,q) - 2pqx_q(p,q) - (1+p^2)x_p(p,q) = 0.$$

  Now we can use the relation in (4) to get:

  $$\begin{aligned}&-(1+q^2)\Phi_{qq}(p,q) - 2pq\Phi_{pq}(p,q) - (1+p^2)\Phi_{pp}(p,q) = 0\\ \Longrightarrow\ &(1+q^2)\Phi_{qq}(p,q) + 2pq\Phi_{pq}(p,q) + (1+p^2)\Phi_{pp}(p,q) = 0.\end{aligned}$$

  This shows that (3) holds at (p,q).

- Case 2: $(x(p,q), y(p,q)) \in C_u$ but $\exists\, (x_n, y_n) \to (x(p,q), y(p,q))$ with $(x_n, y_n) \notin C_u$

  In this case, there is a neighborhood around $(x(p,q), y(p,q))$ in which there exists a sequence (x_n, y_n) such that $u_{xy}(x_n, y_n) \neq 0$, Let $p_n = u_x(x_n, y_n)$, $q_n = u_y(x_n, y_n)$. This sequence lives in the image of the gradient map, and $(p_n, q_n) \to (p, q)$. We showed in the previous case that our identity holds at each (p_n, q_n). Since our expression for $\Phi(p,q)$ and its derivatives is smooth (as u is smooth), the identity thus also holds in the limit as $(p_n, q_n) \to (p, q)$.

- Case 3: $(x(p,q), y(p,q)) \in C_u$ and $\exists$ an open disc $B \subseteq C_u$ centered at $(x(p,q), y(p,q))$

  In this case we note that by assumption u satisfies the minimal surface equation (1), but $u_{xy}(x,y) = 0$ on B (since $B \subseteq C_u$), so the second term drops out of the minimal surface equation. Thus, on B, $u(x,y)$ satisfies the system

  $$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} = 0, \quad u_{xy} = 0.$$

  If $u_{xy} = 0$, then $\partial_y u_x = 0 \Rightarrow u_x$ depends only on x, so $u_x = f'(x) \Rightarrow u = f(x) + h(y)$. Similarly, $u_y = g'(y) \Rightarrow u = g(y) + \tilde{h}(x)$, and combining both, we get:

  $$u(x,y) = f(x) + g(y)$$

  Thus, the condition $u_{xy} = 0$ implies that $u(x,y) = f(x) + g(y)$ locally. We substitute this into the first equation:

  $$(1 + (g')^2)f'' + (1 + (f')^2)g'' = 0.$$

  Separating variables:

  $$\frac{f''}{1 + (f')^2} = -\frac{g''}{1 + (g')^2} = a,$$

  for some constant a. Integrating:

  $$f'(x) = \tan(ax + b), \quad g'(y) = \tan(-ay + d),$$

  so

$$f(x) = -\frac{1}{a}\ln(\cos(ax + b)) + c, \quad g(y) = \frac{1}{a}\ln(\cos(-ay + d)) + e.$$

  We note for the expression of $g(y)$ that $\cos(-\theta) = \cos(\theta)$, and replace d by $-d$ to get:

$$f(x) = -\frac{1}{a}\ln(\cos(ax + b)) + c, \quad g(y) = \frac{1}{a}\ln(\cos(ay + d)) + e.$$

  Hence,

  $$u(x,y) = \frac{1}{a}\ln\left(\frac{\cos(ay + d)}{\cos(ax + b)}\right) + k,$$

for constants a, b, d, k. We compute:

$$p = u_x = \tan(ax + b), \quad q = u_y = \tan(-ay + d),$$

and invert:

$$x(p, q) = \frac{\arctan(p) - b}{a}, \quad y(p, q) = \frac{-\arctan(q) + d}{a}.$$

Thus, we find:

$$\begin{aligned}
\Phi_{pp}(p, q) = x_p(p, q) &= \frac{1}{a(1 + p^2)}, \\
\Phi_{qq}(p, q) = x_q(p, q) &= -\frac{1}{a(1 + q^2)}, \\
\Phi_{pq}(p, q) = x_q(p, q) &= 0.
\end{aligned}$$

Let's substitute these values into equation (3) to see if these values satisfy the linear PDE. We have:

$$\begin{aligned}
(1 + q^2) \cdot \left(-\frac{1}{a(1 + q^2)}\right) + 2pq \cdot 0 + (1 + p^2) \cdot \left(\frac{1}{a(1 + p^2)}\right) = 0 \\
\iff -\frac{1}{a} + \frac{1}{a} = 0,
\end{aligned}$$

which is clearly true, and thus our PDE is satisfied at (p, q).

We will now prove the converse direction. Suppose $\Phi(p, q)$ is a solution of the linear PDE. We have the inverse Legendre function given by:

$$\mathcal{L}^{-1}(\Phi) = p\Phi_p + q\Phi_q - \Phi$$

We now interpret $p = p(x, y)$, $q = q(x, y)$ implicitly via:

$$x = \Phi_p(p, q), \quad y = \Phi_q(p, q),$$

and we reconstruct $u(x, y)$ via the inverse Legendre transform:

$$u(x, y) := \mathcal{L}^{-1}(\Phi)(x, y) = x \cdot \Phi_p(p, q) + y \cdot \Phi_q(p, q) - \Phi(p, q).$$

We want to show that defining a u in this manner gives us a solution of the minimal surface equation (1).

As preliminary work, we will justify that the Legendre transform is an involution, meaning applying it twice brings you back to the original function (when invertibility holds).

We want to verify that if $\Phi(p, q) = x(p, q)\, p + y(p, q)\, q - u(x(p, q), y(p, q))$, then applying the Legendre transform again to Φ brings us back to u, i.e.:

$$\mathcal{L}(\mathcal{L}(u))(s, t) = u(s, t)$$

We again denote:

- $p = u_x(x, y)$, $q = u_y(x, y)$
- $\Phi(p, q) = xp + yq - u(x, y)$

- $x = x(p,q)$, $y = y(p,q)$, the inverse of the gradient map $G(x,y) = (p,q)$
- $s = \Phi_p(p,q)$, $t = \Phi_q(p,q)$

We now compute:

$$\mathcal{L}(\Phi)(s,t) = sp + tq - \Phi(p,q).$$

But recall that:

$$\Phi_p(p,q) = x(p,q), \quad \Phi_q(p,q) = y(p,q) \Rightarrow s = x, \ t = y.$$

So:

$$\mathcal{L}(\Phi)(s,t) = x\,p + y\,q - \Phi(p,q) = u(x,y),$$

since by the definition of Φ:

$$\Phi(p,q) = xp + yq - u(x,y) \Rightarrow u(x,y) = xp + yq - \Phi(p,q).$$

Thus:

$$\mathcal{L}(\mathcal{L}(u))(s,t) = u(s,t)$$

So we've shown that the Legendre transform is involutive (i.e., $\mathcal{L}^2 = \mathrm{id}$) on the class of functions where the gradient map is invertible. This implies that the Legendre transformation is it's own inverse, $\mathcal{L}^{-1} = \mathcal{L}$. We also have that $\mathcal{L}^2(\Phi) = \Phi$, and so:

$$\Phi = \mathcal{L}^2(\Phi) = \mathcal{L}(\mathcal{L}(\Phi)) = \mathcal{L}(\mathcal{L}^{-1}(\Phi)) = \mathcal{L}(u).$$

Thus, all the identities which we derived from this relation, such as (2) and (4), still hold. Now, by our assumption, Φ satisfies the linear PDE, so equation (3) holds. Since (4) holds as well, we can rewrite the equation as:

$$\begin{aligned} &(1+q^2)y_q(p,q) + 2pqx_q(p,q) + (1+p^2)x_p(p,q) = 0 \\ \implies &-(1+q^2)y_q(p,q) - 2pqx_q(p,q) - (1+p^2)x_p(p,q) = 0. \end{aligned}$$

We multiply this by $u_{xy}(x(p,q), y(p,q))$ to get equation (6), and use the same substitutions as given above equation (6) to get (5). This time, we would like to divide by $x_q(p,q)$, which is equal to (since $\Phi_p(p,q) = x(p,q)$ still holds) $\Phi_{pq}(p,q)$. Just as in the forward direction, we face a technical problem: Φ_{pq} might be zero, so we again split into three exhaustive cases (just like we did for the forward direction with the set C_u).

Define the set:

$$K_\Phi = \{(p,q) : \Phi_{pq}(p,q) = 0\},$$

and distinguish the following cases:

- Case 1: $(p,q) \notin K_\Phi$, i.e., $\Phi_{pq}(p,q) = x_q(p,q) \neq 0$

  In this case, we can safely divide both sides of equation (5) by $x_q(p,q)$, giving equation (1), the minimal surface equation. Thus it is satisfied at $(x(p,q), y(p,q))$.

- Case 2: Case 2: $(p,q) \in K_\Phi$, but $\exists (p_n, q_n) \notin K_\Phi \to (p,q)$.

  We again use the same continuity argument as before. At each $(p_n, q_n) \notin K_\Phi$, we know from Case 1 that the minimal surface equation holds at $(x(p_n,q_n), y(p_n,q_n))$. Since $u(x,y)$ is smooth (it's built from smooth Φ), and the minimal surface equation is a smooth function of u's second derivatives, the identity passes to the limit.

- Case 3: $(p,q) \in K_\Phi$, and $\exists$ an open disc $B \subseteq K_\Phi$ centered at (p,q). Similar to the third case as before, we see that on B the system

  $$(1+q^2)\Phi_{qq}(p,q) + (1+p^2)\Phi_{pp}(p,q) = 0, \quad \Phi_{pq} = 0$$

  is satisfied. By the same logic as earlier, this implies that $\Phi = f(p) + g(q)$ for some functions f, g. We substitute this into the first equation:

  $$(1+q^2)g''(q) = a = -(1+p^2)f''(p)$$

  Seperating variables:

  $$\Rightarrow f''(p) = -a/(1+p^2), \quad g''(q) = a/(1+q^2)$$

  Integrating:

  $$\Phi_p(p,q) = f'(p) = -a\arctan(p) + b,$$
  $$\Phi_q(p,q) = g'(q) = a\arctan(q) + c$$

  We know from (4) that $x = \Phi_p(p,q), y = \Phi_q(p,q)$. This implies:

  $$x = f'(p) = -a\arctan(p) + b,$$
  $$y = g'(q) = a\arctan(q) + c$$

  We can invert to express p and q as functions of x and y:

  $$p = \tan\left(\frac{b-x}{a}\right), \quad q = \tan\left(\frac{y-c}{a}\right)$$

  We have the parametric form of u as a function of p and q, by writing:

  $$u(x,y) = x\,p + y\,q - \Phi(p,q).$$

  So now we treat u as a function defined on the parameter space (p,q), via:

  $$u(p,q) = x(p,q)\,p + y(p,q)\,q - \Phi(p,q).$$

  This implies that:

  $$u_x(x,y) = p(x,y) = \tan\left(\tfrac{b-x}{a}\right),$$
  $$u_y(x,y) = q(x,y) = \tan\left(\tfrac{y-c}{a}\right).$$

  Computing the second order derivatives gives

  $$u_{xy}(x,y) = 0$$
  $$u_{xx}(x,y) = -\frac{1}{a}\sec^2\left(\frac{b-x}{a}\right)$$

$$u_{yy}(x,y) = \frac{1}{a}\sec^2\left(\frac{y-c}{a}\right)$$

The implications of $u_{xy}(x,y) = 0$ have already been considered when we created the set C_u. We now plug in the values of u_{xx} and u_{yy} in the minimal surface equation to see if it holds (the second term again drops out, $u_{xy} = 0$):

The first term is

$$(1+u_y^2)u_{xx} = \left(1+\tan^2\left(\frac{y-c}{a}\right)\right)\cdot\left(-\frac{1}{a}\sec^2\left(\frac{b-x}{a}\right)\right).$$

We use the identity

$$1+\tan^2\theta = \sec^2\theta$$

to get:

$$(1+u_y^2)u_{xx} = \sec^2\left(\frac{y-c}{a}\right)\cdot\left(-\frac{1}{a}\sec^2\left(\frac{b-x}{a}\right)\right) = -\frac{1}{a}\sec^2\left(\frac{y-c}{a}\right)\sec^2\left(\frac{b-x}{a}\right).$$

Similarly the second term is:

$$(1+u_x^2)u_{yy} = \left(1+\tan^2\left(\frac{b-x}{a}\right)\right)\cdot\frac{1}{a}\sec^2\left(\frac{y-c}{a}\right) = \sec^2\left(\frac{b-x}{a}\right)\cdot\frac{1}{a}\sec^2\left(\frac{y-c}{a}\right).$$

Adding both:

$$(1+u_y^2)u_{xx} + (1+u_x^2)u_{yy} = -\frac{1}{a}\sec^2(y_c)\sec^2(x_b) + \frac{1}{a}\sec^2(x_b)\sec^2(y_c) = 0$$

(where $x_b = \frac{b-x}{a}$, $y_c = \frac{y-c}{a}$).

Thus, we find that the minimal surface equation (1) indeed holds $\square$