MnT ULTIMATE Recap

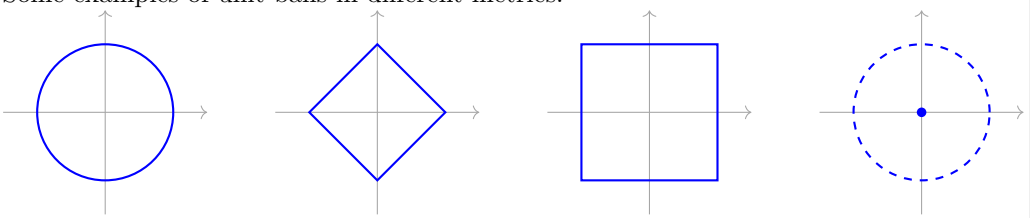
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Recap HW 2

A metric space consists of a non-empty set X together with a function $d : X \times X \to \mathbb{R}$ such that the following hold:
1) $\forall\ x, y \in X,\ d(x, y) \geq 0$; and $d(x, y) = 0 \iff x = y$
2) (Symmetry) $\forall\ x, y \in X,\ d(y, x) = d(x, y)$
3) (Triangle inequality) $\forall x, y, z \in X,\ d(x, z) \leq d(x, y) + d(y, z)$

Some examples of unit balls in different metrics:



Euclidean (d_2) Taxicab (d_1) Sup (d_∞) Discrete (d_0)

For the same set X, you can have several different d_X defined on it.

Definition 5.23 *A subset S of a metric space (X, d) is bounded if* $\exists\, x_0 \in X$ and $K \in \mathbb{R}$ such that $d(x,\, x_0) \leqslant K$ for all $x \in S$.

Definition 5.24 *If S is a non-empty bounded subset of a metric space with metric d, then the diameter of S is* $\sup\{d(x, y) : x, y \in S\}$. *The diameter of the empty set is 0.*

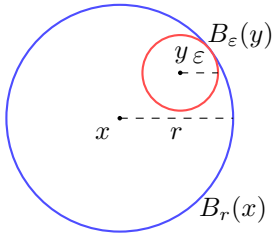
Proposition 5.26 Let (X, d) be a metric space. If $A_1, A_2, \ldots, A_n \subseteq X$ are bounded, then

$$\bigcup_{i=1}^{n} A_i \quad \text{is bounded;}$$

i.e., the union of any finite number of bounded subsets of a metric space is bounded.

Definition 5.27 Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. The open ball of radius r centred on x_0 is

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}.$$



Proposition 5.31 Given an open ball $B_r(x)$ and a point $y \in B_r(x)$, there exists $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq B_r(x)$.

Definition 5.32 $U \subseteq X$ is open in X if for every $x \in U$ there exists $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subseteq U$.

Proposition 5.37 A map $f : X \to Y$ of metric spaces is continuous iff $f^{-1}(U)$ is open in X whenever U is open in Y.

Proposition 5.39 If $U_1, \ldots, U_m$ are open in X, then so is $\bigcap_{i=1}^{m} U_i$.

Recap HW 3

Definition 6.1 A subset V of a metric space X is closed in X if $X \setminus V$ is open in X
So anything you already know about open sets gives you something about closed sets for free. One strategy, if a closed set is giving you trouble, is to take the complement and try the problem again from there.

Some examples of closed sets:

- $[a, b], (-\infty, 0], \{0\}, \{1, 1/2, 1/3, \ldots\} \cup \{0\}$ in $\mathbb{R}$.
- Closed unit disc $\{(x, y) : x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$.
- Any subset of a discrete metric space.
- $\{f \in C([0, 1]) : f(1) = 0\}$ is closed in the sup metric.

Important Propositions:
Prop. 6.3: Finite unions of closed sets are closed.
Prop. 6.4: Arbitrary intersections of closed sets are closed.
Prop. 6.5: $\emptyset$ and X are closed.
Remark: Sets are nothing like doors. A set can be neither open nor closed ($[0,1)$ in $\mathbb{R}$), or both (X itself).

Most closed-set properties are interchangeable with open ones by taking appropriate complements: for example 'pre-image of open is open' translates to
Prop. 6.6: A map $f : X \to Y$ is continuous $\iff$

$$f^{-1}(V) \text{ is closed in } X \quad \text{whenever } V \text{ is closed in } Y.$$

Definition 6.7 The *closure* of $A \subseteq X$ is the set of points $x \in X$ such that

$$\forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset.$$

We add all points that are 'arbitrarily close' to A.

Examples 6.8:

- $\overline{(0,1)} = [0,1]$.
- Closure of an open ball is the corresponding closed ball.
- If $A \subset \mathbb{R}$ is bounded and non-empty, then $\sup A, \inf A \in \overline{A}$.

Proposition 6.11 For $A, B \subseteq X$: (a) $A \subseteq \overline{A}$; (b) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; (c) A closed $\iff A = \overline{A}$; (d) $\overline{\overline{A}} = \overline{A}$; (e) $\overline{A}$ is closed; (f) $\overline{A}$ is the smallest closed set containing A.
Part (b) tends to be the useful one, and it's cheap: if you can spot an inclusion, you get an inclusion of closures for nothing. A few of this week's exercises have one side that falls out of (b), with the actual work sitting in the other direction.

Definition 6.9. $A \subseteq X$ is *dense* in X if $\overline{A} = X$. *Examples:* $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$.

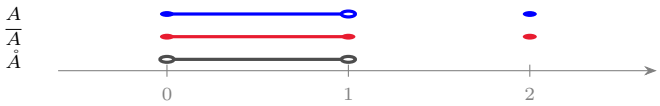
From here on it's Wednesday's lecture.

Definition 6.15 x is a *limit point* of A in X if for every $\varepsilon > 0$ there's a point of A in $B_\varepsilon(x)$ *other than* x *itself*, i.e. $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset$.
Compare with Definition 6.7: the only change is that x doesn't get to count as its own witness. So a point of A may or may not be a limit point of A, but it is always a point of closure.
Prop. 6.18 $\overline{A}$ is A together with all its limit points. **Prop. 6.17** A is closed iff it contains all of its limit points.

Definition 6.19 The *interior* $\mathring{A}$ is the set of $a \in A$ with $B_\varepsilon(a) \subseteq A$ for some $\varepsilon > 0$.
Prop. 6.21 is the mirror image of 6.11: $\mathring{A} \subseteq A$, $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}$, A open iff $A = \mathring{A}$, and $\mathring{A}$ is the largest open set inside A.

Definition 6.22 The *boundary* is $\partial A = \overline{A} \setminus \mathring{A}$.
Prop. 6.24 $x \in \partial A$ iff every $B_\varepsilon(x)$ meets both A and $X \setminus A$. This version is often easier to work with, since it treats A and its complement symmetrically

Here is one example that has all of it at once. Take $A = [0,1) \cup \{2\}$ in $\mathbb{R}$.



So $\overline{A} = [0,1] \cup \{2\}$, $\mathring{A} = (0,1)$, $\partial A = \{0,1,2\}$, and the limit points are $[0,1]$. Note 2 is in A and in $\overline{A}$ but is not a limit point, and 1 is a limit point without being in A. Those two points are roughly what separates Definition 6.15 from Definition 6.7.

Definition 6.25 $(x_n) \to x$ if for every $\varepsilon > 0$ there is N with $x_n \in B_\varepsilon(x)$ for all $n \geq N$. Limits are unique (**Prop. 6.26**, and the proof is 5.5 plus one line), convergent sequences are Cauchy (**Prop. 6.28**), and if (y_n) lives in Y and converges to a then $a \in \overline{Y}$ (**Prop. 6.29**).

Definition 6.31 d_1, d_2 on X are *topologically equivalent* if they make exactly the same subsets open.
Definition 6.33 They are *Lipschitz equivalent* if there are constants $h, k > 0$ with $h\, d_1(x,y) \leq d_2(x,y) \leq k\, d_1(x,y)$ for all x, y.
Prop. 6.34 Lipschitz equivalent $\Rightarrow$ topologically equivalent. The converse is false.

On $\mathbb{R}^n$ the metrics d_1, d_2, d_∞ are all Lipschitz equivalent, which is why the balls look different but the open sets don't. The discrete metric is not equivalent to any of them, and on $C[a,b]$ the sup metric and the L^1 metric are not equivalent either.

Recap HW 4

Definition 7.1 A *topological space* is a pair $(X, \mathcal{T})$ where X is a non-empty set and $\mathcal{T}$ is a family of subsets of X, called open sets, satisfying:
1. $X, \varnothing \in \mathcal{T}$,
2. the intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$,
3. the union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

A topology is like a "rulebook" telling us which subsets of X count as open; these open sets are the framework for continuity.

Proposition 7.2 A set $U \subseteq X$ (topological space)(non-empty!!!🔥) is open iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subseteq U$.
(Note that this is the replacement for "every point of U has a ball inside U")

My favourites, the examples:

- **7.3** any myetric space (X, d) gives a topology $\mathcal{T}_d$, its d-open sets. A space arising this way is *metrizable*.
- **7.4** the *discrete* topology: every subset is open.
- **7.5** the *indiscrete* topology: $\{\varnothing, X\}$ and nothing else.
- **7.7** the Sierpinski space $\mathbb{S} = \{0, 1\}$ with $\{\varnothing, \{1\}, \{0, 1\}\}$.
- **7.9** the *co-finite* topology: $\varnothing$, together with every U whose complement is finite. (This one is super interesting, showed up in my exam)

As far as I remember, Sutherland gives a very interesting analogy regarding the difference between metric-open and topology-open in the textbook. In a metric space you are a nightclub bouncer with a list of criteria to check. In a topological space you are a doorkeeper with a guest list: if the set is on the list, it's open, and there is nothing to verify. Parallels to Copas, as tempting as they are to make, are excluded from the scope of this course (but I invite the reader to exercise their own creativity).

Finer vs. coarser. Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set, $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Equivalently, $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.
Always:

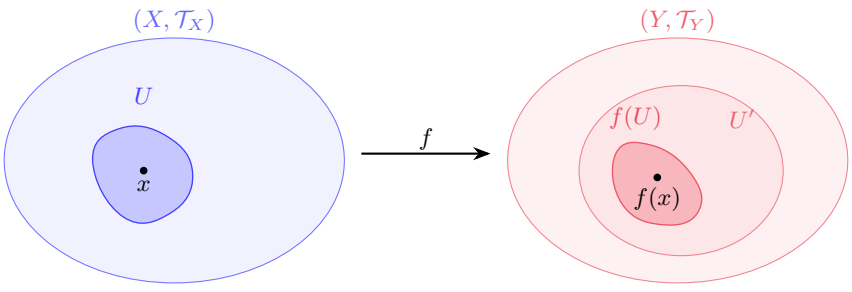
$$\text{Indiscrete topology} \subseteq \mathcal{T} \subseteq \text{Discrete topology}.$$

Two topologies on the same set need not be comparable at all, which is exactly 7.2.

Definition 8.1 A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if

$$U \in \mathcal{T}_Y \quad \Rightarrow \quad f^{-1}(U) \in \mathcal{T}_X.$$

For the billionth time, pre-image of open is open 🗣️
Definition 8.2 f is continuous at $x \in X$ if for every open $U' \subseteq Y$ with $f(x) \in U'$, there exists an open $U \subseteq X$ with $x \in U$ and $f(U) \subseteq U'$. Equivalent: f is continuous everywhere iff it is continuous at each point.



Prop. 8.4 composites of continuous maps are continuous.

Prop. 8.6 identities and constants are continuous, anything out of a discrete space is continuous, anything into an indiscrete space is continuous.

Definition 8.9 A *basis* for $\mathcal{T}$ is a subfamily $\mathcal{B} \subseteq \mathcal{T}$ such that every open set is a union of sets from $\mathcal{B}$. In a metric space the open balls are a basis (that was 5.13), and in $\mathbb{R}$ the open intervals are.

Prop. 8.12 To check f is continuous it is enough to check $f^{-1}(B)$ is open for every B in *some* basis for $\mathcal{T}_Y$.

Definition 8.7 A *homeomorphism* is a bijection $f : X \to Y$ with f and f^{-1} both continuous. Equivalently U is open in X iff $f(U)$ is open in Y, so it is a relabelling that carries the whole rulebook across.

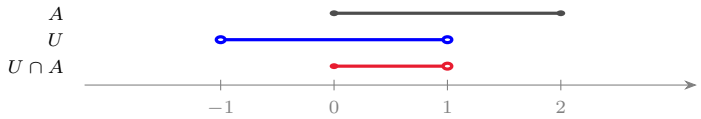
(A property preserved by every homeomorphism is called a *topological invariant*, and Hausdorffness below is one.)

Chapter 9 has stuff you have seen before: V is closed iff $X \setminus V$ is open (**Def. 9.1**), $\varnothing$ and X are closed, finite unions and arbitrary intersections of closed sets are closed (**Prop. 9.4**), and f is continuous iff pre-images of closed sets are closed (**Prop. 9.5**). All proved by taking complements.

Definition 10.3 For $\varnothing \neq A \subseteq X$, the *subspace* (induced, relative) topology is

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}.$$

So "open in A" and "open in X" are now different! Simple example below



$A = [0, 2]$ inside $\mathbb{R}$: the half-open $[0, 1)$ is open in A, even though it is nowhere near open in $\mathbb{R}$.

Prop. 10.4 the inclusion $i : A \hookrightarrow X$ is continuous. **Cor. 10.5** restrictions of continuous maps are continuous. **Prop. 10.6** for $g : Z \to A$, g is continuous iff $i \circ g : Z \to X$ is.

Definition 11.3 X is *Hausdorff* if for any two distinct $x, y \in X$ there are disjoint open U, V with $x \in U$, $y \in V$. Alternatively you can remember that distinct points can be *housed off* from one another, which is probably even more unforgettable in a Russian accent.

Prop. 11.4 in a Hausdorff space limits of sequences are unique. **Prop. 11.5** every metrizable space is Hausdorff (take $\varepsilon = d(x, y)/2$ and use 5.5). **Example 11.6** an infinite set with the co-finite topology is not Hausdorff, since any two non-empty open

sets have finite complements and so must meet. (Thus it is not metrizable either, which is the first time we can prove a space is *not* secretly a metric space 👀)

Recap HW 5

Definition 12.1 A topological space X is *connected* if there does not exist a continuous map from X onto a two-point discrete space.

Definition 12.2 A *partition* $\{A, B\}$ of a topological space X is a pair of non-empty subsets A, B of X such that $X = A \cup B$, $A \cap B = \varnothing$, and both A and B are open in X.

Proposition 12.3 A topological space is connected iff it admits no partition.

Corollary 12.4 A topological space X is connected iff the only subsets of X which are both open and closed in X are $X, \varnothing$.

Definition 12.6 A non-empty subset A of a topological space X is connected if A with the subspace topology is connected according to Definition 12.1. Conventionally we regard the empty set as being connected.

Theorem 12.8 Any connected subspace S of $\mathbb{R}$ is an interval.

Theorem 12.10 Any interval I in $\mathbb{R}$ is connected.

Proposition 12.11 Suppose that $f : X \to Y$ is a continuous map of topological spaces and that X is connected. Then $f(X)$ is connected.

Definition 12.20 For points x, y in a topological space X, a *path* in X from x to y is a continuous map $f : [0, 1] \to X$ such that $f(0) = x$, $f(1) = y$.

Definition 12.21 A topological space X is *path-connected* if any two points of X can be joined by a path in X.

Proposition 12.23 Any path-connected space X is connected.

Recap HW 6

The idea here is that a compact space behaves like a finite set, as far as continuous functions can tell. Compactness is what lets you upgrade something true near every point into something true on the whole space.

Definition 13.3 A family $\{U_i : i \in I\}$ of subsets of X is a *cover* of $A \subseteq X$ if $A \subseteq \bigcup_{i \in I} U_i$.

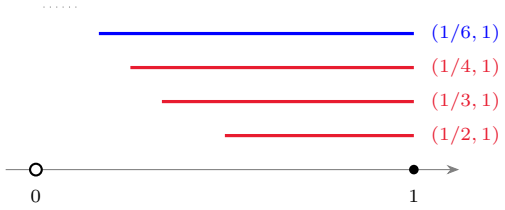
Definition 13.4 A *subcover* of $\{U_i : i \in I\}$ is a subfamily $\{U_j : j \in J\}$ with $J \subseteq I$ which still covers A. It is a *finite subcover* when J is finite.

Definition 13.5 The cover is an *open cover* when each U_i is open in X

Definition 13.6 $A \subseteq X$ is *compact* if *every* open cover of A has a finite subcover.

Remember that you can be given an arbitrary open cover and you have to produce a finite subfamily of *that* cover. It is not the same as "A has some finite open cover", which every subset of every space has, since $\{X\}$ is one.

Example 13.7 $(0,1) \subseteq \mathbb{R}$ is not compact. Cover it by $U_n = (1/n, 1)$ for $n \geq 2$.



Every $x \in (0,1)$ sits in U_n once $n > 1/x$, so this really is a cover. But a finite subfamily has a largest index N and thus only covers $(1/N, 1)$, and the left end of the interval escapes.

Example 13.8 (a) Any finite subset of a space is compact
(b) Any space with the co-finite topology is compact

Theorem 13.9 Heine-Borel my goat: Any closed bounded interval $[a,b] \subseteq \mathbb{R}$ is compact.

Important stuff

- **Prop. 13.10** a compact subset C of a metric space is bounded
- **Cor. 13.11** so any compact subset of $\mathbb{R}^n$ is bounded
- **Prop. 13.12** a compact subset C of a *Hausdorff* space is closed
- **Cor. 13.13** so any compact subset of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.

Proposition 13.15 If $f : X \to Y$ is continuous and X is compact then $f(X)$ is compact. "The continuous image of a compact space is compact."

Corollary 13.16 Compactness is a topological property.

Corollary 13.17 Any continuous map from a compact space to a metric space is bounded: the image is compact, and compact subsets of metric spaces are bounded by 13.10.

Corollary 13.18 A continuous $f : C \to \mathbb{R}$ on a compact C *attains* its bounds, i.e. $\sup f(C)$ and $\inf f(C)$ are values of f. **Cor. 13.19** is the $[a,b]$ case, which is the extreme value theorem

Uhhh ways of building compact things:

- **Prop. 13.20** a closed subset of a compact space is compact
- **Thm. 13.21** $X \times Y$ is compact iff both X and Y are
- **Thm. 13.22 (ok this is the true Heine-Borel)** any closed bounded $C \subseteq \mathbb{R}^n$ is compact

Definitions!

Definition 13.14 $A \subseteq X$ is *relatively compact* in X if $\overline{A}$ is compact, closure taken in X. So $(0,1)$ is relatively compact in $\mathbb{R}$ but not in $(0,1)$.

Definition 13.23 $f : X \to Y$ between metric spaces is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ which works at every point at once

Prop. 13.26 a continuous bijection from a compact space to a Hausdorff space is a homeomorphism

Bolzano-Weierstrass (Thm. 4.19): every bounded real sequence has a convergent subsequence.

Definition 14.2 $S \subseteq \mathbb{R}$ is *sequentially compact* if every sequence in S has a subsequence converging to a point *of* S. **Definition 14.7** same sentence for a metric space X, and **14.8** for a subset with the subspace metric.

Theorem 14.10 and in fact a metric space is compact iff it is sequentially compact. Who could've guessed

How to build stuff that is *not* sequentially compact:

Prop. 14.11 if $B_\varepsilon(x)$ contains x_n for infinitely many n, for every $\varepsilon > 0$, then some subsequence converges to x. Taking the contrapositive, **Cor. 14.13**: if (x_n) has no convergent subsequence then every point x has a radius ε_x with $B_{\varepsilon_x}(x)$ containing only finitely many terms.

Example 14.14 in $X = (0,1]$ the sequence $(1/n)$ has no convergent subsequence, because the only candidate limit is not in X

Recap HW 7

Definition 14.7 A metric space X is *sequentially compact* if every sequence in X has at least one subsequence converging to a point *of* X.

Definition 14.8 A non-empty $A \subseteq (X,d)$ is sequentially compact if it satisfies 14.7 with the subspace metric d_A. The empty set counts as sequentially compact by convention.

Theorem 14.10 A metric space is compact iff it is sequentially compact.

Prop. 14.11 If for every $\varepsilon > 0$ the ball $B_\varepsilon(x)$ contains x_n for infinitely many n, then (x_n) has a subsequence converging to x.

Cor. 14.13 Contrapositive: if (x_n) has no convergent subsequence, then every $x \in X$ has some radius $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x)$ containing only finitely many terms.

Theorem 14.15 Any compact subset of a metric space is sequentially compact.

Definition 14.19 $N \subseteq X$ is an *ε-net* for X if $\{B_\varepsilon(x) : x \in N\}$ covers X.

Example 14.20 The integer lattice points are a 1-net for the plane, and in fact an ε-net for any $\varepsilon > 1/\sqrt{2}$.

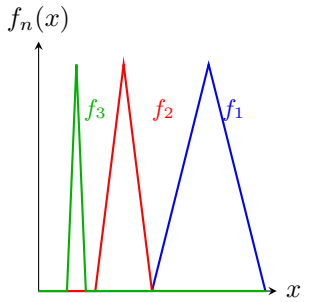
Prop. 14.21 A sequentially compact metric space has a *finite* ε-net for every $\varepsilon > 0$.

Theorem 14.22 Every sequentially compact metric space is compact.

Example 14.23 The moving bump. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2^n}, \\ 2^{n+1}(x - \frac{1}{2^n}), & \frac{1}{2^n} \le x \le \frac{1}{2^n} + \frac{1}{2^{n+1}}, \\ 2^{n+1}(\frac{1}{2^{n-1}} - x), & \frac{1}{2^n} + \frac{1}{2^{n+1}} \le x \le \frac{1}{2^{n-1}}, \\ 0, & \frac{1}{2^{n-1}} \le x \le 1. \end{cases}$$

A bump of height 1 sliding towards 0. In the sup metric $d_\infty(f_m, f_n) = 1$ whenever $m \neq n$, so no subsequence is Cauchy and $B = \{f_n\} \subseteq C[0,1]$ is not sequentially compact.



bumps sliding left as $n \to \infty$

These same functions are used in Example 16.7, where they converge to 0 pointwise but not uniformly.

Pointwise vs uniform.

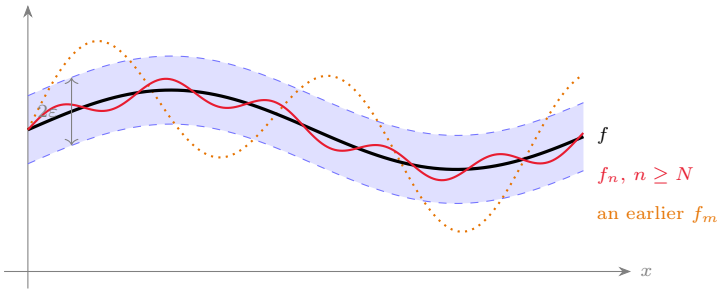
Definition 16.1 $(f_n) \to f$ *pointwise* on D if $f_n(x) \to f(x)$ for each $x \in D$.

Definition 16.3 $(f_n) \to f$ *uniformly* on D if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge N$ *and all* $x \in D$.

Pointwise lets you choose a different N_x at every point, uniform makes one N work

across the whole domain at once.

Draw the ribbon of vertical width 2ε centred on the graph of f. Uniform convergence says: however thin you make the ribbon, from some N onwards the *entire* graph of f_n is inside it.



Example 16.2 $f_n(x) = x^n$ on $[0,1]$ tends pointwise to 0 on $[0,1)$ and to 1 at $x = 1$. Not uniform: take $\varepsilon_0 = 1/2$ and $x = 2^{-1/N}$, so that $x^N = 1/2$ exactly.

Also note that throwing away the bad point doesn't help, $x^n \to 0$ is still not uniform on $[0,1)$. On $[0,a]$ with $a < 1$ it is, because a^n bounds everything.

Proposition 16.4 Put $M_n = \sup_{x \in D} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on D iff M_n exists for all large n and $M_n \to 0$.

So the procedure is: pointwise limit f first, then M_n, then $\lim M_n$. Finding the sup is usually just $f_n' = 0$ plus a check that it's a max.

Example 16.5 $f_n(x) = n^2 x(1-x)^n$ on $[0,1]$: max at $x = 1/(n+1)$, $M_n = n^{n+2}/(n+1)^{n+1} \to \infty$, not uniform!

Example 16.6 $f_n(x) = xe^{-nx^2}$ on $[0,1]$: max at $x = (2n)^{-1/2}$, $M_n = (2en)^{-1/2} \to 0$, uniform!

Theorem 16.9 (Cauchy's criterion) (f_n) converges uniformly on D iff it is *uniformly Cauchy* (Def. 16.8): for every $\varepsilon > 0$ there is N with $|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq N$ and all $x \in D$.

The point of it is that f doesn't have to be known in advance.

Theorem 16.10 If each $f_n : (a,b) \to \mathbb{R}$ is continuous at c and $f_n \to f$ uniformly, then f is continuous at c. **Cor. 16.11** is the same thing on $[a,b]$.

Cor. 16.12 Backwards: if the pointwise limit of continuous functions is discontinuous, the convergence was not uniform. **Example 16.13** $1/(1+nx)$ on $[0,1]$ has pointwise limit 1 at 0 and 0 elsewhere. What 16.10 is really doing is licensing a swap of two limits,

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) \;=\; \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Definition 17.2 A metric space X is *complete* if every Cauchy sequence in X converges to a point of X.

Prop. 17.6 A complete subspace Y of a metric space X is closed in X. **Prop. 17.7** A closed subspace of a complete metric space is complete.

Recap HW 8

Definition 17.2 A metric space X is *complete* if every Cauchy sequence in X converges (to a point of X).

Proposition 17.6 A complete subspace Y of a metric space X is closed in X.

Proposition 17.7 A closed subspace Y of a complete metric space X is complete.

Proposition 17.9 Any compact metric space X is complete.

Definition 17.18 Given any self-map $f : S \to S$ of a set S, a *fixed point* of f is a point $p \in S$ such that $f(p) = p$.

Definition 17.19 For given positive real numbers α and K, a function $f : D \to \mathbb{R}$ satisfies a *Lipschitz condition of order* α on D, with constant K, if

$$|f(x) - f(y)| \leq K|x - y|^\alpha \quad \text{for all } x, y \in D.$$

Definition 17.24 Let (X, d) be a metric space. A map $f : X \to X$ is a *contraction* if for some constant $K < 1$ we have $d(f(x), f(y)) \leq K\, d(x, y)$ for all x, y in X.

Theorem 17.26 (Banach's fixed point theorem) If $f : X \to X$ is a contraction of a complete metric space X then f has a unique fixed point p in X.