MnT Hints & Recap HW 7

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Recap

Definition 14.7 A metric space X is *sequentially compact* if every sequence in X has at least one subsequence converging to a point *of* X.

Definition 14.8 A non-empty $A \subseteq (X, d)$ is sequentially compact if it satisfies 14.7 with the subspace metric d_A. The empty set counts as sequentially compact by convention.

Theorem 14.10 A metric space is compact iff it is sequentially compact.

Prop. 14.11 If for every $\varepsilon > 0$ the ball $B_\varepsilon(x)$ contains x_n for infinitely many n, then (x_n) has a subsequence converging to x.

Cor. 14.13 Contrapositive: if (x_n) has no convergent subsequence, then every $x \in X$ has some radius $\varepsilon_x > 0$ with $B_{\varepsilon_x}(x)$ containing only finitely many terms.

Theorem 14.15 Any compact subset of a metric space is sequentially compact.

Definition 14.19 $N \subseteq X$ is an *ε-net* for X if $\{B_\varepsilon(x) : x \in N\}$ covers X.

Example 14.20 The integer lattice points are a 1-net for the plane, and in fact an ε-net for any $\varepsilon > 1/\sqrt{2}$.

Prop. 14.21 A sequentially compact metric space has a *finite* ε-net for every $\varepsilon > 0$.

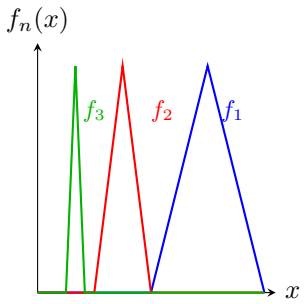
Theorem 14.22 Every sequentially compact metric space is compact.

Example 14.23 The moving bump. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{2^n}, \\ 2^{n+1}(x - \frac{1}{2^n}), & \frac{1}{2^n} \le x \le \frac{1}{2^n} + \frac{1}{2^{n+1}}, \\ 2^{n+1}(\frac{1}{2^{n-1}} - x), & \frac{1}{2^n} + \frac{1}{2^{n+1}} \le x \le \frac{1}{2^{n-1}}, \\ 0, & \frac{1}{2^{n-1}} \le x \le 1. \end{cases}$$

A bump of height 1 sliding towards 0. In the sup metric $d_\infty(f_m, f_n) = 1$ whenever $m \neq n$, so no subsequence is Cauchy and $B = \{f_n\} \subseteq C[0,1]$ is not sequentially

compact.



bumps sliding left as $n \to \infty$

These same functions are used in Example 16.7, where they converge to 0 pointwise but not uniformly.

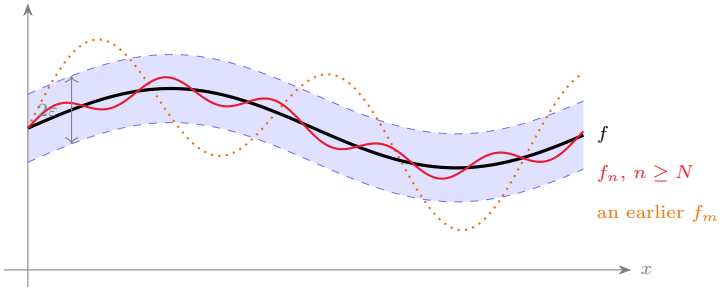
Pointwise vs uniform.

Definition 16.1 $(f_n) \to f$ *pointwise* on D if $f_n(x) \to f(x)$ for each $x \in D$.

Definition 16.3 $(f_n) \to f$ *uniformly* on D if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ *and all* $x \in D$.

Pointwise lets you choose a different N_x at every point, uniform makes one N work across the whole domain at once.

Draw the ribbon of vertical width 2ε centred on the graph of f. Uniform convergence says: however thin you make the ribbon, from some N onwards the *entire* graph of f_n is inside it.



Example 16.2 $f_n(x) = x^n$ on $[0, 1]$ tends pointwise to 0 on $[0, 1)$ and to 1 at $x = 1$. Not uniform: take $\varepsilon_0 = 1/2$ and $x = 2^{-1/N}$, so that $x^N = 1/2$ exactly.

Also note that throwing away the bad point doesn't help, $x^n \to 0$ is still not uniform on $[0, 1)$. On $[0, a]$ with $a < 1$ it is, because a^n bounds everything.

Proposition 16.4 Put $M_n = \sup_{x \in D} |f_n(x) - f(x)|$. Then $f_n \to f$ uniformly on D iff M_n exists for all large n and $M_n \to 0$.

So the procedure is: pointwise limit f first, then M_n, then $\lim M_n$. Finding the sup is usually just $f_n' = 0$ plus a check that it's a max.

Example 16.5 $f_n(x) = n^2x(1-x)^n$ on $[0,1]$: max at $x = 1/(n+1)$, $M_n = n^{n+2}/(n+1)^{n+1} \to \infty$, not uniform!

Example 16.6 $f_n(x) = xe^{-nx^2}$ on $[0,1]$: max at $x = (2n)^{-1/2}$, $M_n = (2en)^{-1/2} \to 0$, uniform!

Theorem 16.9 (Cauchy's criterion) (f_n) converges uniformly on D iff it is *uniformly Cauchy* (Def. 16.8): for every $\varepsilon > 0$ there is N with $|f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq N$ and all $x \in D$.

The point of it is that f doesn't have to be known in advance.

Theorem 16.10 If each $f_n : (a,b) \to \mathbb{R}$ is continuous at c and $f_n \to f$ uniformly, then f is continuous at c. **Cor. 16.11** is the same thing on $[a,b]$.

Cor. 16.12 Backwards: if the pointwise limit of continuous functions is discontinuous, the convergence was not uniform. **Example 16.13** $1/(1+nx)$ on $[0,1]$ has pointwise limit 1 at 0 and 0 elsewhere. What 16.10 is really doing is licensing a swap of two limits,

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) \;=\; \lim_{n\to\infty}\lim_{x\to c} f_n(x).$$

Definition 17.2 A metric space X is *complete* if every Cauchy sequence in X converges to a point of X.

Prop. 17.6 A complete subspace Y of a metric space X is closed in X. **Prop. 17.7** A closed subspace of a complete metric space is complete.

Hints

Problem 14.2

Think about what it means for a metric space not to be bounded. If for every $x_0 \in X$ and every $K > 0$ there's always some point farther than K from x_0, can you build a sequence that keeps going farther and farther away?

Problem 14.3

You already know that a sequentially compact space gives you convergent subsequences. If a sequence converges in a closed subset, where must its limit lie? See Corollary 6.30 👀

Problem 14.4

Try taking a limit point x of S. By the definition of closure, you can find a sequence of points in S that get arbitrarily close to x.

Half of this is free: $S \subseteq \overline{S}$ always holds, so the only inclusion with anything in it is $\overline{S} \subseteq S$.

Problem 14.9

This one requires some creativity so I won't spoil it, just keep taking convergent subsequences of convergent subsequences smartly from both spaces :)

Fix a metric on $X \times Y$ before you start!

Problem 14.11

This is 13.11 from last week with sequences instead of covers. The V_n are non-empty, so

pick $x_n \in V_n$ for each n, and nestedness tells you that for a fixed n, every term x_m with $m \geq n$ is already in V_n. So a single convergent subsequence, plus Cor. 6.30 applied once for each n, lands the limit in all of them at once.

Problem 16.3
Pointwise limit f first, then $M_n = \sup |f_n - f|$ by Prop. 16.4, then $\lim M_n$. The sup is $f_n' = 0$.

Problem 16.4
Haven't we already seen this one in analysis lol

(a) Fix one single $n \geq N$ and triangle-inequality it: $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$. You don't need all the f_n here, one of them is enough, and ε is yours to pick, $\varepsilon = 1$ is fine.

(b) Uniform convergence only tells you something for $n \geq N$; the functions $f_1, \ldots, f_{N-1}$ are each bounded but by their own constants. There are finitely many of them tho, so a max over those together with the bound from (a) gives the single K.