MnT Hints & Recap HW 6

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October 5, 2026

Recap

The idea here is that a compact space behaves like a finite set, as far as continuous functions can tell. Compactness is what lets you upgrade something true near every point into something true on the whole space.

Definition 13.3 A family $\{U_i : i \in I\}$ of subsets of X is a *cover* of $A \subseteq X$ if $A \subseteq \bigcup_{i\in I} U_i$.

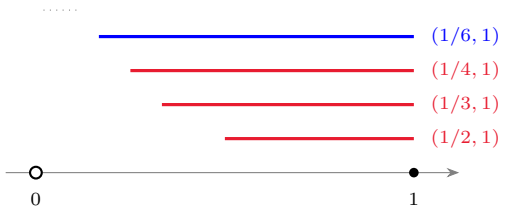
Definition 13.4 A *subcover* of $\{U_i : i \in I\}$ is a subfamily $\{U_j : j \in J\}$ with $J \subseteq I$ which still covers A. It is a *finite subcover* when J is finite.

Definition 13.5 The cover is an *open cover* when each U_i is open in X 🤯

Definition 13.6 $A \subseteq X$ is *compact* if *every* open cover of A has a finite subcover.

Remember that you can be given an arbitrary open cover and you have to produce a finite subfamily of *that* cover. It is not the same as "A has some finite open cover", which every subset of every space has, since $\{X\}$ is one.

Example 13.7 $(0, 1) \subseteq \mathbb{R}$ is not compact. Cover it by $U_n = (1/n, 1)$ for $n \geq 2$.



Every $x \in (0, 1)$ sits in U_n once $n > 1/x$, so this really is a cover. But a finite subfamily has a largest index N and thus only covers $(1/N, 1)$, and the left end of the interval escapes.

Example 13.8 (a) Any finite subset of a space is compact
(b) Any space with the co-finite topology is compact

Theorem 13.9 Heine-Borel my goat: Any closed bounded interval $[a, b] \subseteq \mathbb{R}$ is compact. 🔥

Important stuff

- **Prop. 13.10** a compact subset C of a metric space is bounded
- **Cor. 13.11** so any compact subset of $\mathbb{R}^n$ is bounded
- **Prop. 13.12** a compact subset C of a *Hausdorff* space is closed
- **Cor. 13.13** so any compact subset of $\mathbb{R}^n$ is closed in $\mathbb{R}^n$.

Proposition 13.15 If $f : X \to Y$ is continuous and X is compact then $f(X)$ is compact. "The continuous image of a compact space is compact."

Corollary 13.16 Compactness is a topological property.

Corollary 13.17 Any continuous map from a compact space to a metric space is bounded: the image is compact, and compact subsets of metric spaces are bounded by 13.10.

Corollary 13.18 A continuous $f : C \to \mathbb{R}$ on a compact C *attains* its bounds, i.e. $\sup f(C)$ and $\inf f(C)$ are values of f. **Cor. 13.19** is the $[a, b]$ case, which is the extreme value theorem

Uhhh ways of building compact things:

- **Prop. 13.20** a closed subset of a compact space is compact
- **Thm. 13.21** $X \times Y$ is compact iff both X and Y are
- **Thm. 13.22 (ok this is the true Heine-Borel)** any closed bounded $C \subseteq \mathbb{R}^n$ is compact

Definitions!

Definition 13.14 $A \subseteq X$ is *relatively compact* in X if $\overline{A}$ is compact, closure taken in X. So $(0, 1)$ is relatively compact in $\mathbb{R}$ but not in $(0, 1)$.

Definition 13.23 $f : X \to Y$ between metric spaces is *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ which works at every point at once

Prop. 13.26 a continuous bijection from a compact space to a Hausdorff space is a homeomorphism

Bolzano-Weierstrass (Thm. 4.19): every bounded real sequence has a convergent subsequence.

Definition 14.2 $S \subseteq \mathbb{R}$ is *sequentially compact* if every sequence in S has a subsequence converging to a point *of* S. **Definition 14.7** same sentence for a metric space X, and **14.8** for a subset with the subspace metric.

Theorem 14.10 and in fact a metric space is compact iff it is sequentially compact. Who could've guessed

How to build stuff that is *not* sequentially compact:

Prop. 14.11 if $B_\varepsilon(x)$ contains x_n for infinitely many n, for every $\varepsilon > 0$, then some subsequence converges to x. Taking the contrapositive, **Cor. 14.13**: if (x_n) has no convergent subsequence then every point x has a radius ε_x with $B_{\varepsilon_x}(x)$ containing only finitely many terms.

Example 14.14 in $X = (0, 1]$ the sequence $(1/n)$ has no convergent subsequence, because the only candidate limit is not in X

Hints

Problem 13.4

Recall the Heine-Borel theorem for subsets of $\mathbb{R}^n$. Think about how to test if a set is closed (are all limit points included?) and bounded (does it fit in a large enough ball?). For sets in $\mathbb{R}^2$, consider using preimages of closed intervals under continuous functions to show closedness.

Problem 13.10

We've shown in a previous homework that if $V \subseteq X$ is closed in X, then $V \cap A$ is closed in $(A, \mathcal{T}_A)$ (A being a subspace of X). Is $C \cap C'$ closed in X?

Problem 13.11

First try to prove exercise 13.6, the finite intersection property. What contradiction would arise if the infinite intersection were empty?

(Hint within a hint: look at the open complements.)

Problem 13.14

Propositions 10.13, 10.12, exercise 5.17 👀

Problem 13.15

The exercise literally tells you how to do it step by step lmao, what i will say is the contradiction will come from the openness of the covering sets.