MnT Hints & Recap HW 4

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Recap

Definition 7.1 A *topological space* is a pair $(X, \mathcal{T})$ where X is a non-empty set and $\mathcal{T}$ is a family of subsets of X, called open sets, satisfying:
1. $X, \varnothing \in \mathcal{T}$,
2. the intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$,
3. the union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

A topology is like a "rulebook" telling us which subsets of X count as open; these open sets are the framework for continuity.

Proposition 7.2 A set $U \subseteq X$ (topological space)(non-empty!!!🔥) is open iff for every $x \in U$, there exists an open set U_x such that $x \in U_x \subseteq U$.
(Note that this is the replacement for "every point of U has a ball inside U")

My favourites, the examples:

- **7.3** any myetric space (X, d) gives a topology $\mathcal{T}_d$, its d-open sets. A space arising this way is *metrizable*.
- **7.4** the *discrete* topology: every subset is open.
- **7.5** the *indiscrete* topology: $\{\varnothing, X\}$ and nothing else.
- **7.7** the Sierpinski space $\mathbb{S} = \{0, 1\}$ with $\{\varnothing, \{1\}, \{0, 1\}\}$.
- **7.9** the *co-finite* topology: $\varnothing$, together with every U whose complement is finite. (This one is super interesting, showed up in my exam)

As far as I remember, Sutherland gives a very interesting analogy regarding the difference between metric-open and topology-open in the textbook. In a metric space you are a nightclub bouncer with a list of criteria to check. In a topological space you are a doorkeeper with a guest list: if the set is on the list, it's open, and there is nothing to verify. Parallels to Copas, as tempting as they are to make, are excluded from the scope of this course (but I invite the reader to exercise their own creativity).

Finer vs. coarser. Given two topologies $\mathcal{T}_1, \mathcal{T}_2$ on the same set, $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Equivalently, $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.
Always:

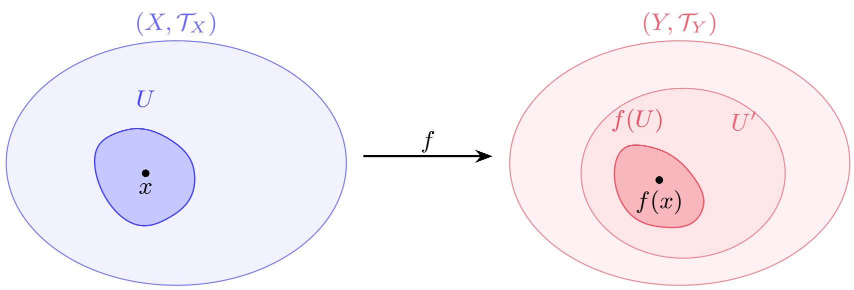
$$\text{Indiscrete topology} \subseteq \mathcal{T} \subseteq \text{Discrete topology}.$$

Two topologies on the same set need not be comparable at all, which is exactly 7.2.

Definition 8.1 A map $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if

$$U \in \mathcal{T}_Y \quad \Rightarrow \quad f^{-1}(U) \in \mathcal{T}_X.$$

For the billionth time, pre-image of open is open 🗣️
Definition 8.2 f is continuous at $x \in X$ if for every open $U' \subseteq Y$ with $f(x) \in U'$, there exists an open $U \subseteq X$ with $x \in U$ and $f(U) \subseteq U'$. Equivalent: f is continuous everywhere iff it is continuous at each point.



Prop. 8.4 composites of continuous maps are continuous.

Prop. 8.6 identities and constants are continuous, anything out of a discrete space is continuous, anything into an indiscrete space is continuous.

Definition 8.9 A *basis* for $\mathcal{T}$ is a subfamily $\mathcal{B} \subseteq \mathcal{T}$ such that every open set is a union of sets from $\mathcal{B}$. In a metric space the open balls are a basis (that was 5.13), and in $\mathbb{R}$ the open intervals are.

Prop. 8.12 To check f is continuous it is enough to check $f^{-1}(B)$ is open for every B in *some* basis for $\mathcal{T}_Y$.

Definition 8.7 A *homeomorphism* is a bijection $f : X \to Y$ with f and f^{-1} both continuous. Equivalently U is open in X iff $f(U)$ is open in Y, so it is a relabelling that carries the whole rulebook across.

(A property preserved by every homeomorphism is called a *topological invariant*, and Hausdorffness below is one.)

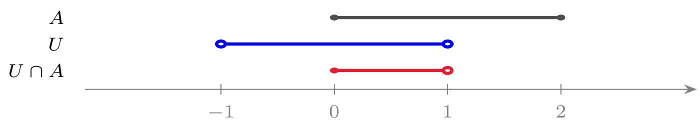
Chapter 9 has stuff you have seen before: V is closed iff $X \setminus V$ is open (**Def. 9.1**), $\varnothing$ and X are closed, finite unions and arbitrary intersections of closed sets are closed

(**Prop. 9.4**), and f is continuous iff pre-images of closed sets are closed (**Prop. 9.5**). All proved by taking complements.

Definition 10.3 For $\varnothing \neq A \subseteq X$, the *subspace* (induced, relative) topology is

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}.$$

So "open in A" and "open in X" are now different! Simple example below



$A = [0, 2]$ inside $\mathbb{R}$: the half-open $[0, 1)$ is open in A, even though it is nowhere near open in $\mathbb{R}$.

Prop. 10.4 the inclusion $i : A \hookrightarrow X$ is continuous. **Cor. 10.5** restrictions of continuous maps are continuous. **Prop. 10.6** for $g : Z \to A$, g is continuous iff $i \circ g : Z \to X$ is.

Definition 11.3 X is *Hausdorff* if for any two distinct $x, y \in X$ there are disjoint open U, V with $x \in U$, $y \in V$. Alternatively you can remember that distinct points can be *housed off* from one another, which is probably even more unforgettable in a Russian accent.

Prop. 11.4 in a Hausdorff space limits of sequences are unique. **Prop. 11.5** every metrizable space is Hausdorff (take $\varepsilon = d(x, y)/2$ and use 5.5). **Example 11.6** an infinite set with the co-finite topology is not Hausdorff, since any two non-empty open sets have finite complements and so must meet. (Thus it is not metrizable either, which is the first time we can prove a space is *not* secretly a metric space 👀)

Hints
Problem 7.2
If only I had my man Wacław Sierpiński to help me out with this

Problem 7.3
If you did Problem 7.2, maybe you can use those topologies to show that either the intersection or union would not necessarily always work.

Problem 7.4
Just the three axioms. Be a little careful with an infinite union:)

Problem 7.5
If only I had my man Augustus De Morgan and his laws to help me out with this

Guys did you know De Morgan was actually born in India?

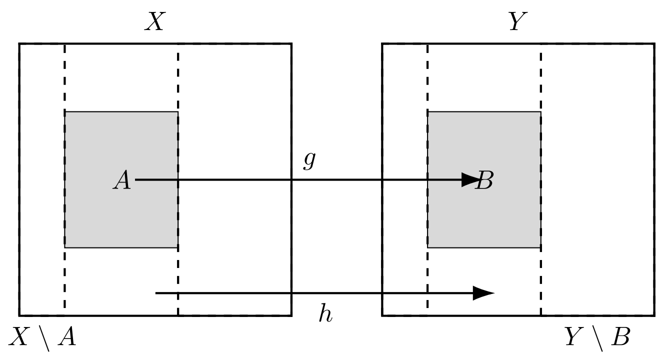
(Is $\varnothing$ in the topology by the finiteness condition? 👀)

Problem 8.2
Evidently (hopefully) we have to use the 'pre-image of open is open' idea here somewhere,

combine this with proposition 3.14 in one of the directions. For the other direction, use the double inclusion lemma to show that a pre-image is a union of some sets.

Problem 10.5
Book hint fire icl 🔥

Problem 10.10
Draw sketches!



Also Proposition 10.6:

$$\begin{tikzcd} B \arrow[r, "f^{-1}|_B"] \arrow[dr, swap, "i\circ(f^{-1}|_B)"] & A \arrow[d, "i"] \\ & X \end{tikzcd}$$

Remember that ultimately, g and h are both just functions induced by f on disjoint subsets of X, Y.

Problem 11.3
Since you have been given Definition 11.3, what proof method seems the most appropriate here?