MnT Hints & Recap HW 3

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Recap

Definition 6.1 A subset V of a metric space X is closed in X if $X \setminus V$ is open in X 🤯🔥
So anything you already know about open sets gives you something about closed sets for free. One strategy, if a closed set is giving you trouble, is to take the complement and try the problem again from there.

Some examples of closed sets:

- $[a, b], (-\infty, 0], \{0\}, \{1, 1/2, 1/3, \dots\} \cup \{0\}$ in $\mathbb{R}$.
- Closed unit disc $\{(x, y) : x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$.
- Any subset of a discrete metric space.
- $\{f \in C([0, 1]) : f(1) = 0\}$ is closed in the sup metric.

Important Propositions:
Prop. 6.3: Finite unions of closed sets are closed.
Prop. 6.4: Arbitrary intersections of closed sets are closed.
Prop. 6.5: $\emptyset$ and X are closed.
Remark: Sets are nothing like doors. A set can be neither open nor closed ($[0, 1)$ in $\mathbb{R}$), or both (X itself).

Most closed-set properties are interchangeable with open ones by taking appropriate complements: for example 'pre-image of open is open' translates to
Prop. 6.6: A map $f : X \to Y$ is continuous $\iff$

$$f^{-1}(V) \text{ is closed in } X \quad \text{whenever } V \text{ is closed in } Y.$$

Definition 6.7 The *closure* of $A \subseteq X$ is the set of points $x \in X$ such that

$$\forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset.$$

We add all points that are 'arbitrarily close' to A.

Examples 6.8:

- $\overline{(0,1)} = [0,1]$.
- Closure of an open ball is the corresponding closed ball.
- If $A \subset \mathbb{R}$ is bounded and non-empty, then $\sup A, \inf A \in \overline{A}$.

Proposition 6.11 For $A, B \subseteq X$: (a) $A \subseteq \overline{A}$; (b) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$; (c) A closed $\iff A = \overline{A}$; (d) $\overline{\overline{A}} = \overline{A}$; (e) $\overline{A}$ is closed; (f) $\overline{A}$ is the smallest closed set containing A.
Part (b) tends to be the useful one, and it's cheap: if you can spot an inclusion, you get an inclusion of closures for nothing. A few of this week's exercises have one side that falls out of (b), with the actual work sitting in the other direction.

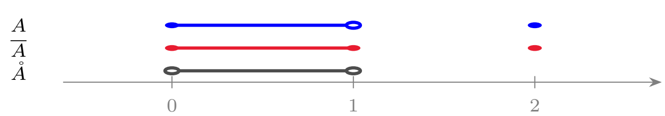
Definition 6.9. $A \subseteq X$ is *dense* in X if $\overline{A} = X$. *Examples:* $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$.

From here on it's Wednesday's lecture.

Definition 6.15 x is a *limit point* of A in X if for every $\varepsilon > 0$ there's a point of A in $B_\varepsilon(x)$ *other than x itself*, i.e. $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset$.
Compare with Definition 6.7: the only change is that x doesn't get to count as its own witness. So a point of A may or may not be a limit point of A, but it is always a point of closure.
Prop. 6.18 $\overline{A}$ is A together with all its limit points. **Prop. 6.17** A is closed iff it contains all of its limit points.

Definition 6.19 The *interior* $\mathring{A}$ is the set of $a \in A$ with $B_\varepsilon(a) \subseteq A$ for some $\varepsilon > 0$.
Prop. 6.21 is the mirror image of 6.11: $\mathring{A} \subseteq A$, $A \subseteq B \Rightarrow \mathring{A} \subseteq \mathring{B}$, A open iff $A = \mathring{A}$, and $\mathring{A}$ is the largest open set inside A.
Definition 6.22 The *boundary* is $\partial A = \overline{A} \setminus \mathring{A}$.
Prop. 6.24 $x \in \partial A$ iff every $B_\varepsilon(x)$ meets both A and $X \setminus A$. This version is often easier to work with, since it treats A and its complement symmetrically 💯

Here is one example that has all of it at once. Take $A = [0,1) \cup \{2\}$ in $\mathbb{R}$.



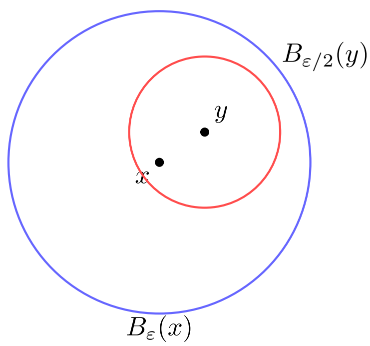
So $\overline{A} = [0,1] \cup \{2\}$, $\mathring{A} = (0,1)$, $\partial A = \{0,1,2\}$, and the limit points are $[0,1]$. Note 2 is in A and in $\overline{A}$ but is not a limit point, and 1 is a limit point without being in A. Those two points are roughly what separates Definition 6.15 from Definition 6.7.

Definition 6.25 $(x_n) \to x$ if for every $\varepsilon > 0$ there is N with $x_n \in B_\varepsilon(x)$ for all $n \geq N$. Limits are unique (**Prop. 6.26**, and the proof is 5.5 plus one line), convergent sequences are Cauchy (**Prop. 6.28**), and if (y_n) lives in Y and converges to a then $a \in \overline{Y}$ (**Prop. 6.29**).

Definition 6.31 d_1, d_2 on X are *topologically equivalent* if they make exactly the same subsets open.
Definition 6.33 They are *Lipschitz equivalent* if there are constants $h, k > 0$ with $h\, d_1(x, y) \leq d_2(x, y) \leq k\, d_1(x, y)$ for all x, y.
Prop. 6.34 Lipschitz equivalent $\Rightarrow$ topologically equivalent. The converse is false.
On $\mathbb{R}^n$ the metrics d_1, d_2, d_∞ are all Lipschitz equivalent, which is why the balls look different but the open sets don't. The discrete metric is not equivalent to any of them, and on $C[a, b]$ the sup metric and the L^1 metric are not equivalent either.

Hints
Problem 5.5
It should be immediately obvious that you want to argue by a contradiction :)

Problem 5.6
Draw a sketch:



A triangle inequality should get you the rest of the way from there. It might be worth doing even if you don't hand it in, since it's the same step that hides inside the proof of Proposition 6.11(d).

Problem 6.6
This may seem a bit unfamiliar, but perhaps you can use Proposition 6.4 here 👀 One option is to write the set as an intersection over $a \in A$ of something simpler, so you only have one kind of closed set left to check.

Problem 6.9
Recall the definition of closure, and remember supremum properties (from analysis!): first show that for all $\varepsilon > 0$ there exists an $a \in A$ such that $a > \sup(A) - \varepsilon$.
Side remark: Is it guaranteed that we have a supremum and an infimum?

Problem 6.10
Since A is bounded, we can use the definition of boundedness to obtain some mathematical

inequality. We also ultimately want some kind of inequality to bound $\overline{A}$. From the open ball definition of closure, maybe obtain another inequality? (The fact that I've used the word inequality like thrice here should be a big hint).

For the diameter half, one of the two inequalities might come cheaply: $A \subseteq \overline{A}$, and you did 5.9 last week.

Problem 6.13
Hmmm I wonder if Proposition 5.30 would help perhaps.

The backward implication is the longer one. One route is to aim for 'pre-image of closed is closed' (Prop. 6.6) and feed $A = f^{-1}(V)$ into the assumption. Proposition 3.14 might come in handy along the way.

Problem 6.14
One inclusion comes cheaply, and it's the one people tend to spend half a page on: each $A_i \subseteq \bigcup A_j$, so 6.11(b) might settle that side in a couple of lines.

The other side is where the finiteness seems to matter. One thing you could try: take x in the closure of the union and suppose x is in none of the $\overline{A_i}$. Each of those hands you an ε_i, and there are only finitely many, so there's something fairly natural to do with them.

If you want to see why finite matters, look at $\bigcup_{q \in \mathbb{Q}} \{q\}$ in $\mathbb{R}$.

Problem 6.15
This is the same cheap inclusion as in 6.14, except the arrow points the other way round, since $\bigcap A_i \subseteq A_j$ for every j. That might well be the whole proof, honestly.

For the counterexample I think two sets are already enough. Try to get them disjoint but almost touching, so the closures pick up the same missing point.

Problem 6.23
There are four parts here and most of them are bookkeeping rather than anything clever. I found Proposition 6.24 easier to work with than Definition 6.22, since it's symmetric in A and $X \setminus A$ and that symmetry is roughly what part (c) is about. Part (b) can be done as a De Morgan argument: complement everything and see what the statement is saying about open sets.

Once you have (c), part (d) should be very short 👀