The alternating compositions of weighted differential operators yield the weights' Wronskian with which constant ?

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What is the Wronskian?

Setup. Take functions
$f, g, h, \ldots$ and their derivatives
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Linear independence

$$\Longleftrightarrow$$

$$\mathrm{Wronsk} \not\equiv 0$$

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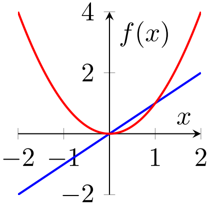

A general N-ary bracket

$$\mathcal{A}_p[w_1, \ldots, w_N](f) \ := \ \sum_{\sigma \in S_N} (-1)^\sigma \, w_{\sigma(1)} \partial_x^p \circ w_{\sigma(2)} \circ \cdots \circ w_{\sigma(N)} \partial_x^p \, (f(x)),$$

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$$\mathcal{A}_p[w_1, \ldots, w_N](f) \;=\; \mathrm{const}(p) \cdot \mathrm{Wronsk}(w_1, \ldots, w_N) \cdot \partial_x^p(f(x)).$$

Problem formulation

$$\mathcal{A}_p[w_1, \ldots, w_N](f) \;=\; \mathrm{const}(p) \cdot \mathrm{Wronsk} \cdot \partial_x^p(f(x))$$

What is const(p)?

Problem formulation

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What is const(p)?

For $p = 1$, $N = 2$: $\mathrm{const}(p) = 1$.

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What is const(p)?

For $p = 1$, $N = 2$: const(p) $= 1$.

p	$N!$	const(p)
1	2	1
2	24	2
3	40320	90

The hunt for const(p)

p	$N!$	const(p)
1	2	1
2	24	2
3	720	90
4	40 320	586 656
5	3 628 800	1 915 103 977 500
6	479 001 600	$\approx 7.886 \times 10^{21}$
7	$\approx 8.7 \times 10^{10}$	$\approx 8.587 \times 10^{34}$

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- const(p) grows extremely quickly with p
- For large values acceleration methods required
- All const(p) values are integers
- Computation up to const($p = 14$) possible on modest hardware, takes ≈ 5 hours
- Values reproduced, corrected and independently verified up to const($p = 7$)

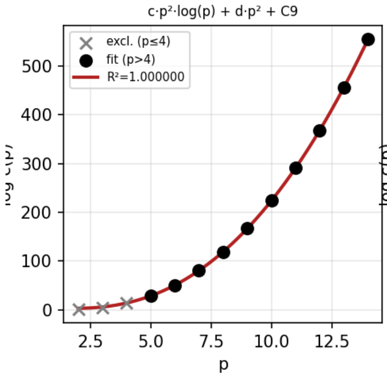
Results

p	$\text{const}(p)$
1	1
2	2
3	90
4	586 656
5	1 915 103 977 500
6	$\approx 7.886 \times 10^{21}$
7	$\approx 8.587 \times 10^{34}$

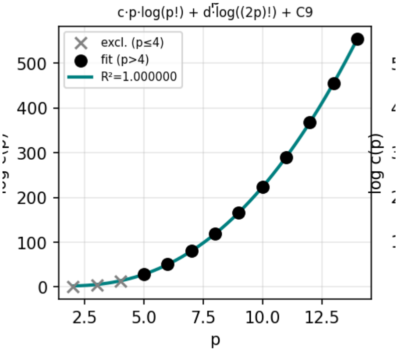
p	$\text{const}(p)$
8	$\approx 4.594 \times 10^{51}$
9	$\approx 2.060 \times 10^{72}$
10	$\approx 1.237 \times 10^{97}$
11	$\approx 1.512 \times 10^{126}$
12	$\approx 5.498 \times 10^{159}$
13	$\approx 8.396 \times 10^{197}$
14	$\approx 7.398 \times 10^{240}$

Growth Rate

Two good fits



(a) $c \cdot p^2 \log p + d \cdot p^2 + \text{const}$



(b) $c \cdot p \log(p!) + d \cdot \log((2p)!) + \text{const}$

Looking further

- **More values.** Push computation to larger p
- **Asymptotic laws.** Pin down the correct growth rate.
- **Prime structure.** Prime decomposition of const(p)
- **Multidimensional case.** Generalisation of the Wronskian identity to d dimensions.
- ***N*-ary brackets in physics.**
  - Lada & Stasheff, `arXiv:hep-th/9209099` (18pp.), A_∞,L_∞-algebras
  - AVK, `arXiv:2510.02145` (8pp.)