The alternating compositions of weighted differential operators yield the weights' Wronskian with which constant ?

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Values & data

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Outline

Setup

const(p)

Reduction to Combinatorics

Values

Growth and Outlook

Prelude: the Wronskian

$$W(w_1, \dots, w_n)(x) = \det \begin{pmatrix} w_1(x) & w_2(x) & \cdots & w_n(x) \\ w_1'(x) & w_2'(x) & \cdots & w_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ w_1^{(n-1)}(x) & w_2^{(n-1)}(x) & \cdots & w_n^{(n-1)}(x) \end{pmatrix}.$$

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Example: the Lie bracket of two vector fields

$$\big[\, w_1\partial_x,\ w_2\partial_x \,\big](f) \;=\; \big(w_1w_2' - w_2w_1'\big)\, f' \;=\; 1 \cdot W(w_1,w_2)\cdot \partial_x f.$$

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$$\mathcal{A}_p[w_1, \ldots, w_N](f) := \sum_{\sigma \in S_N} (-1)^\sigma \, w_{\sigma(1)} \partial_x^p \circ w_{\sigma(2)} \partial_x^p \circ \cdots \circ w_{\sigma(N)} \partial_x^p \, (f),$$

where $(-1)^\sigma$ is the sign of the permutation σ.

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Special inputs for the N-ary Lie bracket

$$[\, a_1, a_2, \ldots, a_N \,] \ := \ \sum_{\sigma \in S_N} (-1)^\sigma \, a_{\sigma(1)} \circ a_{\sigma(2)} \circ \cdots \circ a_{\sigma(N)}$$

over an associative algebra A, with $a_i = w_i \, \partial_x^p$.

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- Total derivatives: Np. Leibniz redistributes, never destroys.
- Antisymmetry forces the counts $k_1, \ldots, k_N$ on the weights to be pairwise distinct: swapping two equal ones fixes the monomial and flips its sign.
- Hence $\sum_i k_i \geq 0 + 1 + \cdots + (N-1) = \frac{N(N-1)}{2}$.

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So the order in f is at most $Np - \frac{N(N-1)}{2}$, and at least p from the innermost ∂_x^p. Demanding strict order p:

$$Np - \frac{N(N-1)}{2} = p \quad \Longleftrightarrow \quad p(N-1) = \frac{N(N-1)}{2}$$

$$\Longleftrightarrow \quad \boxed{N = 2p}$$

The constant $\mathrm{const}(p)$

Theorem (Kiselev [2, 5])

There is a universal constant $\mathrm{const}(p)$*, independent of* f *and of the weights, such that*

$$\mathcal{A}_p[w_1,\ldots,w_N](f) \;=\; \mathrm{const}(p)\cdot \mathrm{Wronsk}(w_1,\ldots,w_N)\cdot \partial_x^p(f).$$

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Central question. What is the value of $\mathrm{const}(p)$?

Known before this work

p	$\mathrm{const}(p)$	
1	1	(Lie bracket)
2	2	(by hand)
3	90	(JETS, symbolic)

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A direct expansion over S_N is already infeasible at $p = 4$ ($8! = 40320$ terms). A more economical strategy is needed for $p \geq 4$.

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Right-hand side

$$\det\begin{pmatrix} 1! & x & \cdots & x^{N-1} \\ 0 & 1! & \cdots & (N-1)x^{N-2} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & (N-1)! \end{pmatrix} = \prod_{k=1}^{N-1} k!.$$

A closed-form expression

For $\sigma \in \Phi_p$ and $k = 1, \ldots, N-1$, set

$$E_k(\sigma) \ := \ \sum_{j=0}^{k-1} \big(\sigma(N-j) - 1\big) \ - \ (k-1)p.$$

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Theorem (Closed form)

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monomial term $\longleftrightarrow$ σ $\longleftrightarrow$ summand

Worked example: $p = 2$

Computing const(2)

Contributing permutations: (1,2,3,4), (1,2,4,3), (1,3,2,4)

$$\mathrm{const}(p)\Big|_{p=2} = \frac{3!\,3!\,2! - 3!\,2!\,2! - 3!\,2!\,2!}{3!\,2!\,1!} = 3! - 2! - 2! = 2.$$

Values of const(p), $1 \le p \le 14$

p	const(p)
1	1
2	2
3	90
4	586 656
5	1 915 103 977 500
6	$\approx 7.886 \times 10^{21}$
7	$\approx 8.587 \times 10^{34}$

p	const(p)
8	$\approx 4.594 \times 10^{51}$
9	$\approx 2.060 \times 10^{72}$
10	$\approx 1.237 \times 10^{97}$
11	$\approx 1.512 \times 10^{126}$
12	$\approx 5.498 \times 10^{159}$
13	$\approx 8.396 \times 10^{197}$
14	$\approx 7.398 \times 10^{240}$

New integer sequence: oeis.org/A392714[1]
Values const(1)-const(7) verified externally by R. Buring

[1] Also permanently available on kianshah.in (under Research).

Growth of const(p)

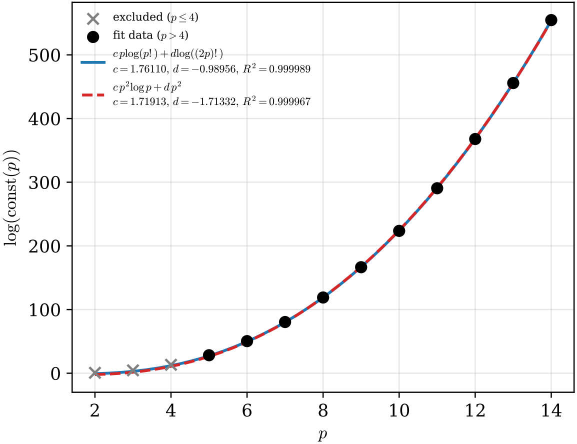

Figure: The two best-fitting growth laws for $\log \mathrm{const}(p)$ (fit over $p > 4$; excluded points $p \leq 4$ marked with a cross).

Refining the fit

The six-parameter raw polynomial

$$\log \operatorname{const}(p) \;\approx\; \alpha\, p^2 \log p + \beta\, p^2 + \gamma\, p \log p + \delta\, p + \epsilon \log p + \mu,$$

fitted over three windows:

Interval	α	β	γ	δ	ϵ	μ	max $\lvert$resid$\rvert$
$[5;14]$	**1.998535**	-2.667511	0.909466	-0.027817	3.023176	2.510459	9.7×10^{-7}
$[6;14]$	**1.998823**	-2.669029	0.920547	-0.052795	3.049267	2.530522	2.3×10^{-7}
$[7;14]$	**1.999002**	-2.669988	0.927923	-0.069907	3.067888	2.543470	4.9×10^{-8}

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$$\alpha \longrightarrow 2$$

Arithmetic structure

$q \setminus p$	**2**	**3**	4	**5**	6	**7**	8	9	10	**11**	12	**13**	14
2	**1**	1	5	2	5	4	17	11	16	7	18	13	17
3	·	**2**	3	1	6	2	2	14	6	5	11	8	3
5	·	1	·	**4**	2	·	·	·	8	6	5	2	2
7	·	·	1	1	·	**6**	5	2	·	·	·	·	13
11	·	·	·	·	1	·	·	2	·	**10**	8	6	4
13	·	·	·	·	·	1	1	1	·	1	·	**12**	11
17	·	·	·	·	·	·	·	1	·	·	·	1	·
19	·	·	·	·	·	·	·	·	1	1	·	·	·

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7	·	·	1	1	·	**6**	5	2	·	·	·	·	13
11	·	·	·	·	1	·	·	2	·	**10**	8	6	4
13	·	·	·	·	·	1	1	1	·	1	·	**12**	11
17	·	·	·	·	·	·	·	1	·	·	·	1	·
19	·	·	·	·	·	·	·	·	1	1	·	·	·

Conjecture (the diagonal): $v_p(\mathrm{const}(p)) = p - 1$ for every prime p.

Conclusion

Done

Binary Lie bracket $\rightarrow N = 2p$ arguments, Wronskian arises. const(p) exact for all $p \leq 14$ (241 digits). OEIS A392714, arXiv:2605.11137.

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In progress

Positivity and the bound $p^2 \log p \leq \log \mathrm{const}(p) \leq 2p^2 \log p$ (O. Zaboronski, private communication); the fits give $\alpha \to 2$, its upper end.

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Open

- Prove $v_p(\mathrm{const}(p)) = p - 1$; the lower bound is established.
- Parity: even and odd counts in Φ_p differ by exactly one.
- The exact growth rate; the values $p \geq 15$.
- Over $\mathbb{R}^d$: arity $\binom{d+k}{d}$ [6], a multivariable generalisation.

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