GEOMETRY AND TOPOLOGY: LECTURE 2

KIAN SHAH
S5578191

"For I am a bear of very little brain, and long words bother me"

— Winnie the Pooh, perhaps on orientable manifolds

Definition of orientability of a manifold

As we have seen the definition of a (real, n-dimensional) manifold, we can now move on to the orientability property of manifolds. Our setup and conventions are as follows: let M^n be a smooth n-dimensional manifold (for our favourite dimension n, and without boundary, for simplicity). We write charts as

$$\varphi : U \subset M \to \tilde{U} \subset \mathbb{R}^n,$$

and a transition map between overlapping charts ϕ and ψ as

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

The Jacobian matrix of a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ at a point x is denoted $DF(x)$ and its determinant $\det(DF(x))$.

With this in mind, we can give a definition for an orientable manifold. Suppose M is covered by charts $\{(U_\alpha, \varphi_\alpha)\}$. For each pair of overlapping charts $\varphi_\alpha, \varphi_\beta$ (i.e., $U_\alpha \cap U_\beta \neq \emptyset$) consider the transition map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

If for every overlap and every point in the overlap we have

$$\det\big(D(\varphi_\beta \circ \varphi_\alpha^{-1})(x)\big) > 0,$$

then we say the atlas is an *oriented atlas.* If a manifold admits such an atlas, we call the manifold *orientable.*

For some intuition, the Jacobian determinant of a transition map measures whether the switch between the two coordinate systems preserves or reverses the handedness of the coordinate basis. Positive determinant = preserves handedness. If no overlap ever flips handedness, we can say "this is the positive orientation" everywhere.

In accordance with the lecture, we define $x-$beautiful coordinates x_α^i and $X-$ugly coordinates $X_\beta^{\prime j}$. With this we give the definition:

Date: November 24, 2025.

Definition If for M^n $\exists$ an atlas $A = \{U_\alpha\}, \alpha \in \mathcal{I}$, such that you have transition functions (reparametrizations):

$$x^i_\alpha \rightleftarrows X'^j_\beta$$

$$\& \; U_\alpha \cap U_\beta \neq \emptyset$$

with $(\forall\ \alpha, \beta | U_\alpha \cap U_\beta \neq \emptyset)$:

$$\det\left(\frac{\partial x^i_\alpha}{\partial X'^j_\beta}\right) > 0,$$

Then M^n is *orientable.*

Examples of manifolds $M_1 = M_2$ with a smooth map $f : M_1 \to M_2$ that is orientation-preserving but not bijective

As asked for in the lecture, we give explicit examples in dimensions 1 and 2. In each case the underlying sets M_1 and M_2 are equal, the Jacobian determinant of f is everywhere positive in local coordinates, yet f fails to be bijective. Finally we explain the generalization to dimension n.

Let

$$M_1 = M_2 = S^1 = \{\, e^{i\theta} : \theta \in \mathbb{R} \,\}.$$

Define

$$f : S^1 \longrightarrow S^1, \qquad f(e^{i\theta}) = e^{i(2\theta)}.$$

Choose the standard coordinate chart

$$\varphi : (0, 2\pi) \to S^1 \setminus \{1\}, \qquad \varphi(\theta) = e^{i\theta}.$$

In this chart, f is represented by the local coordinate map

$$\tilde{f}(\theta) = 2\theta \pmod{2\pi}.$$

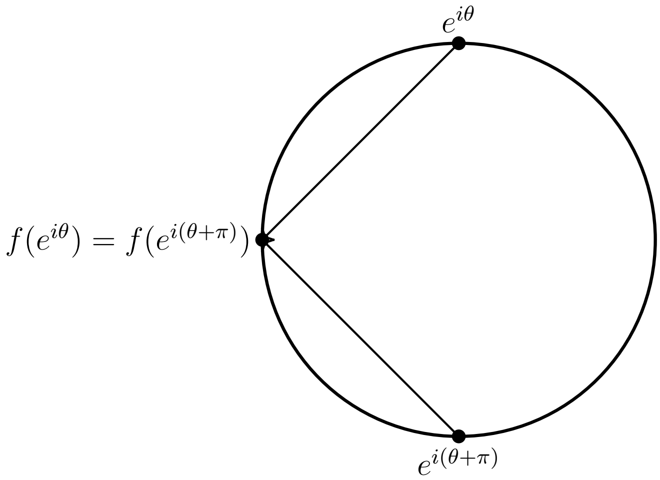
Thus its Jacobian is simply the derivative

$$D\tilde{f}(\theta) = 2 > 0.$$

Thus f is orientation-preserving in every chart. However, f is *not* bijective:

$$f(e^{i\theta}) = f(e^{i(\theta+\pi)}),$$

so it is 2-to-1.



For the 2-dimensional case, consider the torus equivalent for this map. Let

$$M_1 = M_2 = T^2 = S^1 \times S^1.$$

Use angular coordinates $(\theta_1, \theta_2) \in (0, 2\pi)^2$. Define the smooth map

$$f : T^2 \to T^2, \qquad f(\theta_1, \theta_2) = (2\theta_1,\ \theta_2) \pmod{2\pi}.$$

The Jacobian matrix in these coordinates is

$$Df(\theta_1, \theta_2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \det Df = 2 > 0.$$

Thus f is orientation-preserving, but f is not injective:

$$f(\theta_1, \theta_2) = f(\theta_1 + \pi, \theta_2).$$

The pattern for the general n-dimensional case should be evident now. Let $M_1 = M_2 = T^n = (S^1)^n$. Define

$$f(\theta_1, \dots, \theta_n) = (2\theta_1, \theta_2, \dots, \theta_n) \pmod{2\pi}.$$

The Jacobian matrix is diagonal with entries $(2, 1, \dots, 1)$, so

$$\det Df = 2 > 0.$$

Thus f is orientation-preserving in every dimension n, but it remains non-injective since

$$f(\theta_1, \theta_2, \dots, \theta_n) = f(\theta_1 + \pi, \theta_2, \dots, \theta_n).$$

Example: "thick line" with two zeros: a non-Hausdorff manifold

Let

$$X := \big(\mathbb{R} \setminus \{0\}\big) \ \cup\ \{0_1, 0_2\},$$

as a set. We give X the topology generated by

- all ordinary open sets $U \subset \mathbb{R}$ with $0 \notin U$ (viewed as subsets of X); and

- for each $\varepsilon > 0$ the neighborhoods

$$(-\varepsilon, 0) \cup \{0_1\} \cup (0, \varepsilon) \qquad \text{and} \qquad (-\varepsilon, 0) \cup \{0_2\} \cup (0, \varepsilon).$$

The points $x \neq 0$ keep their usual neighborhoods, while 0 is replaced by two distinct points $0_1, 0_2$.

Define $p : X \to E^1 = \mathbb{R}$ as:

$$p(x) = \begin{cases} x, & x \in \mathbb{R} \setminus \{0\}, \\ 0, & x = 0_1 \text{ or } x = 0_2. \end{cases}$$

This function is *surjective* and *not injective* since $p(0_1) = p(0_2) = 0$. Furthermore, p is continuous since the pre-image of any open set is open. The pre-image of a set not containing 0 is evidently open, and the pre-image of a set containing 0 would be the union of two open sets (and thus open). Thus we have a homeomorphism to the Euclidean line, but the manifold is not Hausdorff.

(Big) Examples of manifolds: projective spaces

The real projective plane $\mathbb{RP}^n$ is the set of all points $(a_0, a_1, \cdots, a_n) \sim (\lambda a_0, \lambda a_1, \cdots, \lambda a_n)$ for $a_i \in \mathbb{K}$ (which here is $\mathbb{R}$), and $\lambda \in \mathbb{K} \setminus \{0\}$. This means that we identify tuples from our space with their nonzero scalar multiples.

To see an application of this idea, we write down the equation $a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n = 0$ and multiply both sides by some scalar λ. Although philosophically one can wonder whether or not this is the same equation, mathematically we still get the exact same set of solutions. Thus, such algebraic equations are points of the projective plane, since we are only interested in the coefficients 'modulo' their scalar factor.

Another interpretation of the this space is the following:

$$\mathbb{RP}^n := \text{all 1-dimensional linear subspaces of } \mathbb{R}^{n+1}.$$

A point in $\mathbb{RP}^n$ is a line through the origin:

$$[x] = \lambda x : \lambda \in \mathbb{R} \setminus \{0\}, \qquad x \in \mathbb{R}^{n+1} \setminus \{0\}.$$

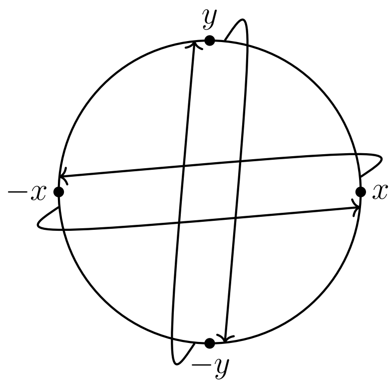
Two vectors represent the same point iff they differ by a nonzero scalar:

$$x \sim y \iff y = \lambda x \quad (\lambda \neq 0).$$

Yet another interpretation is the unit sphere with antipodal identification:

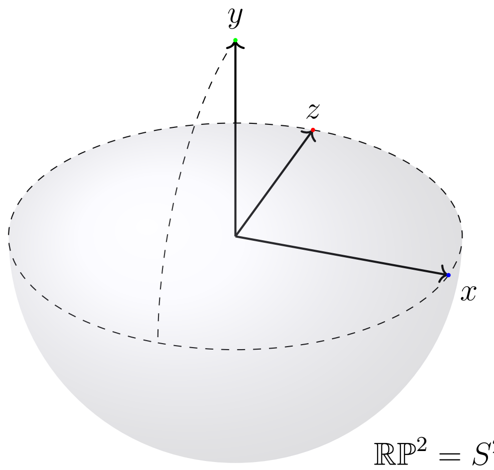
$$\mathbb{RP}^n = S^n/(x \sim -x).$$

We take the n-sphere (set of points unit distance from the origin) and glue every point to its antipodal point.



$\mathbb{RP}^1 = S^1/(x \sim -x)$

Since antipodal points are identified, we can always just consider points from one half of the n-sphere to represent our projective plane. For example, if we take $\mathbb{RP}^2$, we get the set of points on the bottom half of the sphere as so:

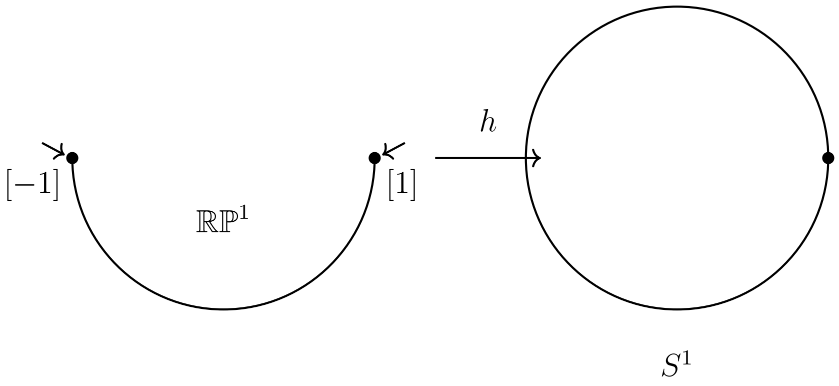


$\mathbb{RP}^2 = S^2/(x \sim -x)$

Naturally, we also have $\mathbb{RP}^1$ as a semi-circle in $\mathbb{R}^2$, which is homeomorphic to $\mathbb{S}^1$. To get the explicit homeomorphism, let $[e^{i\theta}]$ be a point of $\mathbb{RP}^1$ (here $[x]$ is the equivalent class of a point x). Choose the unique $\theta \in [\pi, 2\pi]$ representing its class. Then define

$$h([e^{i\theta}]) = e^{i(2\theta - 2\pi)}.$$

This is a continuous, bijective, orientation-preserving map from the semi-circle with endpoints identified to the full circle.



DIRECT CARTESIAN PRODUCT OF TWO MANIFOLDS

Let M and N be smooth m- and n-dimensional manifolds (can be topological or smooth).

The direct product manifold

$$M \times N$$

is the Cartesian product of sets

$$M \times N = \{(p, q) \mid p \in M, , q \in N\}$$

equipped with the product topology, i.e. the smallest topology for which all projections

$$\pi_M : M \times N \to M, \qquad \pi_N : M \times N \to N$$

are continuous. The product of two manifolds is again a manifold, of dimension $m + n$. As for the charts on $M \times N$, suppose (U, φ) is a chart of M and (V, ψ) is a chart of N. Then

$$(U \times V\,; \varphi \times \psi)$$

is a chart of $M \times N$, where

$$(\varphi \times \psi)(p, q) := \big(\varphi(p), \psi(q)\big) \in \mathbb{R}^{m+n}.$$

These charts form an atlas on $M \times N$.

For example, the standard 2-torus is

$$T^2 = S^1 \times S^1.$$

As a product of manifolds, S^1 is a 1-manifold, so $S^1 \times S^1$ is therefore a 2-manifold. Each factor has the atlas

$$S^1 \setminus (1, 0) \to (0, 2\pi), \quad e^{i\theta} \mapsto \theta.$$

Thus a chart on the torus is simply

$$(\theta, \phi) \in (0, 2\pi) \times (0, 2\pi),$$

which are angular coordinates.

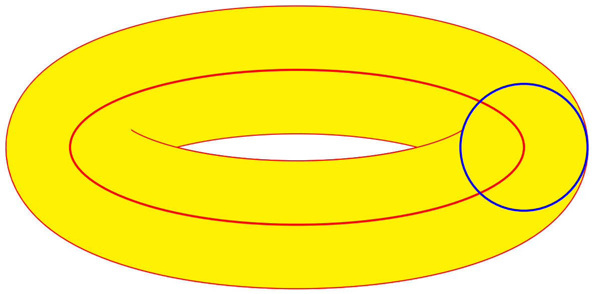
The product structure is hidden when you draw the torus in $\mathbb{R}^3$, but the usual embedding

$$\begin{cases} x = (R + r\cos\phi)\cos\theta, \\ y = (R + r\cos\phi)\sin\theta, \\ z = r\sin\phi, \end{cases} \qquad \theta, \phi \in [0, 2\pi)$$

is literally parameterized by

$$(\theta, \phi) \in S^1 \times S^1.$$

Thus the"doughnut shape" is just the image of the product atlas for $S^1 \times S^1$, which we've encountered earlier in another section as well:



Bonus: gluing along a boundary

A very common way to build new manifolds is by gluing two manifolds with boundary along a common boundary component.

Let M and N be smooth n-manifolds with boundary, and suppose we have a diffeomorphism

$$\varphi : \partial M \longrightarrow \partial N.$$

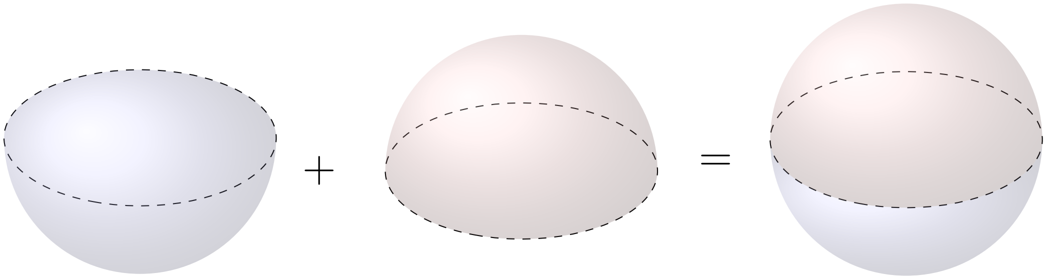
The glued manifold $M \cup_\varphi N$ is obtained by taking the disjoint union $M \sqcup N$ and identifying each point $x \in \partial M$ with $\varphi(x) \in \partial N$. For example, gluing two disks along their boundary gives a sphere. We take two copies of the 2-disk:

$$D^2 = (x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1.$$

Each has boundary $\partial D^2 = S^1$. If we glue their boundaries using the identity map $\mathrm{id} : S^1 \to S^1$, we obtain the 2-sphere:

$$D^2 \cup_{\mathrm{id}} D^2 \cong S^2.$$

We can imagine this by deforming the disk to form a hollow hemisphere (a bowl-like shape).



For a higher-dimensional example, we glue two solid (i.e, filled in) tori along their boundary, the classical ‘torus’ (not filled in). Let $V = D^2 \times S^1$ be a solid torus. Its boundary is the ordinary torus:

$$\partial V = S^1 \times S^1.$$

If you take two solid tori V_1 and V_2, and glue their boundaries by an appropriate diffeomorphism

$$\varphi : \partial V_1 \to \partial V_2,$$

then the resulting manifold is:

$$V_1 \cup_\varphi V_2 \;\cong\; S^3.$$